MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 94 No. 7

MEETING OF 1934 MAY 11

Prof. F. J. M. Stratton, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

Marcel de Kerolyr, Assistant-observateur, Station d'Astrophysique de l'Observatoire de Paris, Forcalquier (B.A.), France ;

B. S. Madhava Rao, M.Sc., Professor at the College of Engineering, Mysore University, Bangalore, India ; and

Cecil Frederick Wickes, 113 Ninth Avenue, Durban, S. Africa.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Ernest Valentine Payne, 17 Carson Road, Dulwich, London, S.E. 21 (proposed by W. H. Steavenson) ; and

Lieutenant E. H. Thompson, R.E., 59th (Field) Coy., R.E., Canterbury (proposed by L. J. Comrie).

Ninety-four presents were announced as having been received since the last meeting.

Dr. Harlow Shapley delivered the George Darwin Lecture,* taking as his subject "Some Structural Features of the Metagalaxy" ; after which the President presented him with the Gold Medal of the Society, which had been awarded to him for his Studies of the Structure and Dimensions of the Galactic System.

* It is hoped to publish the Lecture in the Suppt. number of the *Monthly Notices*.

At the conclusion of the lecture the President said:

As was announced some months ago, the Council of the Society has awarded its Gold Medal this year to Professor Harlow Shapley for his studies of the Structure and Dimensions of the Galactic System. In one sense the thanks of the Society were given to him in advance for the eloquent lecture to which we have all listened with such pleasure. But to-day we are expressing our gratitude to him for much more than for his simple exposition of a vast and complicated subject; by the award of our Medal we want to express our gratitude for the great amount of work of primary importance that Professor Shapley has carried out over a period of years in elucidating one of the major problems of the Universe. I shall not attempt to say more on this subject. The George Darwin lecture has amply freed me from any need to lay before you the grounds of the Council's award. But I may remind you of other aspects of the active astronomical career of our Medallist.

To many here Dr. Shapley is probably best known as a worker in the field of variable stars. His earliest published work was on the elements of a number of eclipsing variables—using observations made at Harvard Observatory. At that time Dr. Shapley was at Princeton Observatory; working with Professor Russell he had developed the theory of the effect on the light curves of eclipsing variables of darkening at the limb. He made early practical applications of this theory to a number of individual stars, and also studied such questions as the effect of the ellipsoidal shape of the components of binary stars; one of his early papers alone gave the orbits of ninety eclipsing variables. Later at Mount Wilson Observatory he turned his attention to the asymmetric light curves of Cepheid variables and the changes of spectral type, putting forward the pulsation theory, which has been the source of much subsequent fruitful discussion. Dr. Shapley also extended the use of the relation discovered by Miss Leavitt, between the mean absolute magnitude and the period of a Cepheid variable, to determine the distances of the stellar clusters; in this we can trace one link between his earlier work on variable stars and his later work on the structure of the Universe. Before turning to that side of his researches I must mention the service that Dr. Shapley rendered to variable star observers during the ten years that he acted as President of the Commission on Variable Stars of the International Astronomical Union. The valuable reports drawn up by him in this capacity must not be neglected in any account of his contributions to astronomy.

But I must turn back to another aspect of Dr. Shapley's work at Mount Wilson. He assisted Dr. Seares in the extension of the North Polar sequence of photographic and photovisual magnitudes—a most suitable task for a future director of the Harvard Observatory. He then turned his attention to the colours and magnitudes of star clusters, and in a series of papers examined such features as their internal structure, their distances and their relation to the Galaxy. From these studies he derived views of the structure and dimensions of the galactic system which were widely different from

those previously held, though in their main features they have since met with general acceptance.

With his record of achievement during seven years on the staff of Mount Wilson Observatory, it was natural that he should be selected in 1921 to succeed to the Directorship of that great centre of astronomical observation and research, the Harvard College Observatory. Of his work there I need not say more than that he has thoroughly maintained a great tradition, and that under his fostering care the observatory has extended even further the wide range of its astronomical activities in both the Northern and the Southern Hemispheres. In many branches of astronomy Harvard still collects a great mass of facts, and discusses their meaning and their bearing on current theories. We have had full illustration of these features in to-day's lecture. A number of those present to-night, when attending the General Assembly of the International Astronomical Union in 1932, welcomed the opportunity of inspecting the Harvard Observatory and saw for themselves the fruits of the great organising power of the Director; they also profited by this same power, directed as it was successfully to the comfort of the visiting astronomers and to the smooth working of an international conference.

Dr. Shapley is a man of many parts. We have read recently in *Nature* of his speed-temperature curve for ants, which shows him to be a careful observer of objects of a very different order of size to galaxies and metagalaxies; such books as *The Source Book in Astronomy* reveal in him a keen historical sense. His earliest published papers in astronomical journals were on astronomy in Horace and Lucretius, and from the latter I may fairly quote a passage descriptive of our Medallist:

> Ergo vivida vis animi pervicit et extra
> processit longe flammantia mœnia mundi
> atque omne immensum peragravit mente animoque
> unde refert nobis victor quid possit oriri
> quid nequeat.

Therefore the living force of his soul gained the day: on he passed far beyond the flaming walls of the world and traversed throughout in mind and spirit the immeasurable universe; whence he returns a conqueror to tell us what can, what cannot, come into being.

Professor Shapley, this Medal which I ask you to accept from the Royal Astronomical Society is an earnest of our deep appreciation of your work, and it carries with it our heartfelt wishes for much further success in the years to come, and for many years of happiness both for yourself and for one of your astronomical colleagues, Mrs. Shapley.

The President then vacated the Chair, and the Meeting was adjourned while Dr. E. P. Hubble, on behalf of the Astronomical Society of the Pacific, presented the Catharine Bruce Gold Medal to Professor Alfred Fowler, to whom it had been awarded "for distinguished services to Astronomy."

The President resumed the Chair, and the last part of the Meeting was devoted to a discussion on The Expansion of the Universe, in which the following Fellows and Associates took part : Professor E. A. Milne, Dr. E. P. Hubble, Abbé G. Lemaître, Dr. Harlow Shapley, Dr. J. H. Oort, Dr. W. H. McCrea.

THE SOLAR LIMB EFFECT.

A. Hunter, B.Sc., D.I.C.

(Communicated by A. Fowler, D.Sc., F.R.S.)

During the past thirty years the equatorial rotation period of the Sun has been under investigation by spectroscopic means at Edinburgh Observatory. By the courtesy of Professor Sampson, two plates representative of those obtained during 1932 have been lent for measurement here. The region photographed is in the red, near the α group of atmospheric absorption, the dispersion obtained with the grating employed being 0·71 A. per mm.* The measurements were subjected to an interesting method of reduction which requires the use of the spectrum of the centre of the solar disc as well as those of the E. and W. limbs. This method incidentally involves the calculation of quantities from which the "limb displacement"—the difference between the wave-length of a Fraunhofer line at the centre of the disc and that at the limb—may be simply derived. In view of the discordance between the values obtained for this quantity at widely separated observatories, it has been thought desirable to record the results obtained here. Accordingly the first part of this communication describes the method of reduction used, and gives the results obtained; the second is devoted to a discussion of the mechanisms hitherto proposed to explain the phenomenon.

The plates measured were taken on 1932 June 15 and June 30, with the Sun high. Each consists of nine spectra—four sets of two exposures to the E. and W. limbs respectively, and a single exposure to the centre of the disc.† Eight separate determinations of the limb shift for each line were therefore possible without assuming the Doppler rotation shift. In all, 34 lines were measured on each exposure : 10 metallic lines, whose increase in wave-length at the Sun's limb was to be measured, and 24 telluric lines, which provided a datum level for the elimination of the rotation shift. A complete list of wave-lengths, sources and intensities of these lines is given in Table I.

The plates used are particularly suited to the determination of the limb shift, since the region photographed is in the red, and what evidence there

* For a full description of the apparatus used in securing the photographs, see Storey, *M.N.*, **92**, 737, 1932.

† For the limb exposures, the light was taken from a point about one-sixteenth of the radius inside the edge, in order to avoid chromospheric interference.

TABLE I

Wave-length *	Source	Intensity	Wave-length *	Source	Intensity
6280·396 A.	Ao	2	6299·231 A.	Ao	3
80·628	Fe	3	6301·517	Fe	7
81·185	Ao	1	02·005	Ao	2
81·961	Ao	2	02·508	Fe	5
82·730	Ao	2	02·771	Ao	2
83·799	Ao	1	05·819	Ao	2
84·545	Ao	0	06·575	Ao	2
87·750	Ao	1	09·896	Ao	2
89·403	Ao	1	10·643	Ao	1
90·224	Ao	2	11·511	Fe	1
90·981	Fe	4	14·244	Ao	0
92·170	Ao	2	14·675	Ni	4
92·967	Ao	3	14·983	Ao	0
95·186	Ao	3	15·822	Fe	1
95·966	Ao	3	18·036	Fe	6
97·808	Fe	5	19·600	Ao	0
98·462	Ao	2	22·701	Fe	4

is points to an increase of displacement with wave-length.† The lines do not show marked variations of character or intensity; settings were therefore easy, and the measurements may be expected to be homogeneous. Moreover, it is well known that the presence of other lines close to a line being measured introduces systematic errors of magnitudes depending on the proximity, intensity and contour of these neighbouring lines; the dispersion used, however, is sufficient to free the region from all imperfectly resolved blends, *i.e.* to give for each line measured enough separation from adjacent lines for an equally strong continuous background to appear on both sides.‡ The use of the spectra of E. and W. limbs rather than polar limbs, which should give no rotation shift, is advantageous since the rate of change of linear velocity is a minimum at the equator, where slight errors in position angle will therefore have least effect. Moreover, the constancy of the total shift provides a useful check on the stability of the instrumental adjustments.

The Hilger L1 micrometer used for the measurements was provided with an eye-piece containing two vertical threads 0·055 mm. apart. This arrangement was found conducive to greater confidence, speed and accuracy in setting than is possible with a single vertical wire, for since the separation of the fiducial threads is less than the width of a metallic line, and more than

* From the *Revision of Rowland's Preliminary Table of Solar Spectrum Wave-lengths.*

† Adams, *Ap. J.*, **31**, 30, 1910; Evershed and Royds, *Kodaikanal Obs. Bull.*, No. 39, 1914.

‡ The importance of this condition has been shown by St. John and Ware, *Ap. J.*, 44, 15, 1916.

that of a telluric line, it is as easy to set on the one as on the other. In order to reduce to a minimum the effect of accidental errors, twelve settings were made on each line. Each spectrum was measured separately, and since the displaced lines were not seen together in the field of view, no question of an unconscious tendency to equalise the red and violet displacements could arise.

Though the method of reduction employed was not devised originally for the purpose of measuring the limb displacement, it shows the effect very clearly, and is in some respects to be preferred to the usual method of direct comparison between adjacent limb and centre spectra.

Consider first the spectrum of the centre of the disc. If, for a grating

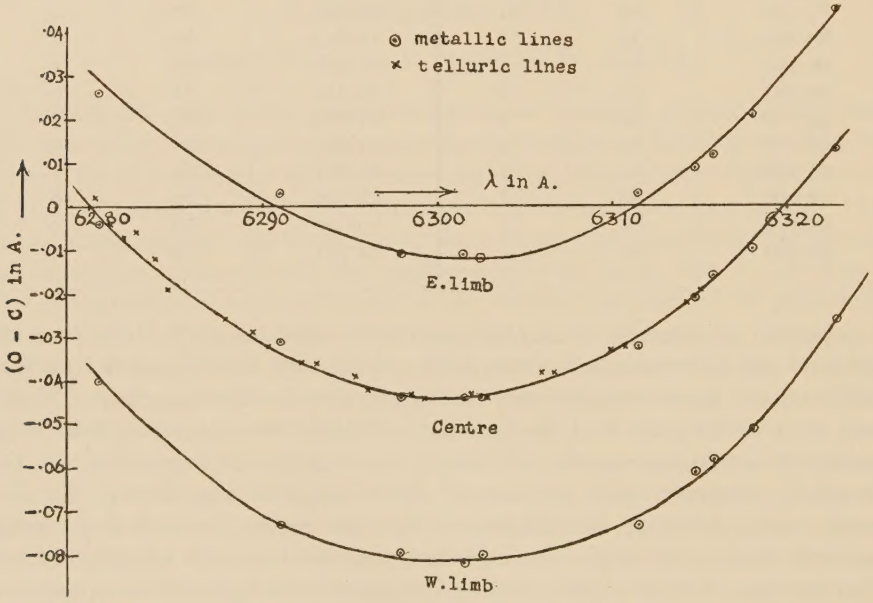

FIG. 1.—1932 *June* 30, *exposures* 7, 8 *and* 9.

spectrum, a linear formula is used to connect the wave-length of a line with its scale reading, it is well known that the wave-lengths (C) thus computed will differ in a regular way from the observed values (O). In an error diagram of the usual type, where the values of (O – C) are plotted against (C), using the revised Rowland wave-lengths in Table I as (O), points corresponding to all the telluric and metallic lines in the "centre" spectrum will lie on a smooth curve (the central curve of fig. 1). A similar procedure carried out for the E. limb exposure will give for the *telluric* lines points lying on the same curve. Points corresponding to the *metallic* lines, which suffer a Doppler shift to the violet, will, however, lie on another curve, parallel to the first but displaced from it by an amount corresponding to the violet shift (upper curve, fig. 1). In the same way, the metallic lines in the W. limb exposure give a similar curve displaced in the opposite direction by the corresponding red shift (lower curve, fig. 1). The distances in angstroms between the outer curves of fig. 1 and the central datum curve

are evidently equal to the mean displacements at the two limbs; whilst the distance of each individual point on an outer curve from the datum curve represents the particular shift suffered by the corresponding line. Thus the shifts are determined not by measuring directly from the nearest telluric line, but by using *all* the telluric lines as standards of reference.

Fig. 1, which is representative of all eight sets of exposures, shows clearly that, neglecting signs, the violet displacements (e) at the E. limb are uniformly less than the red (w) at the W. limb. The process outlined above has been carried out for all the exposures measured, and the results are embodied in Table II, which gives the individual values of e and w for the mean of the four sets of exposures on each day. The inequality between the two shifts is again apparent.

TABLE II

λ approx.	Mean Shift in A.							
	June 15				June 30			
	w	e	$\frac{w-e}{2}$	$\frac{w+e}{2}$	w	e	$\frac{w-e}{2}$	$\frac{w+e}{2}$
6280·6	+·044	−·032	+·038	+·006	+·041	−·034	+·038	+·004
91·0	+·046	−·032	·039	·007	+·044	−·033	·038	·005
97·8	+·043	−·032	·038	·006	+·041	−·032	·036	·004
6301·5	+·045	−·032	·038	·006	+·041	−·033	·037	·004
02·5	+·048	−·032	·040	·008	+·043	−·032	·037	·005
11·5	+·044	−·038	·041	·003	+·040	−·035	·038	·003
14·7	+·045	−·034	·039	·005	+·041	−·031	·036	·005
15·8	+·043	−·032	·038	·006	+·040	−·035	·037	·003
18·0	+·046	−·030	·038	·008	+·041	−·032	·037	·004
22·7	+·046	−·030	+·038	+·008	+·041	−·034	+·038	+·004
Means	+·045	−·032	+·039	+·006	+·041	−·033	+·037	+·004

The discrepancy is consistent with the postulation of a displacement to the red at both limbs, in addition to and independent of the rotation shift. Thus in fig. 2 the dotted lines show qualitatively the calculated positions of a typical line at the two limbs, assuming only a rotation shift. A limb shift to the red of magnitude $+d$ will displace these to the positions shown by the full lines, thus producing measured shifts of w and e. It is easy to show that the true rotation shift is equal to $\pm (w-e)/2$, and that the limb displacement $d=(w+e)/2$. The mean values of these quantities for each line appear in Table II.

The agreement between the individual values of $(w-e)/2$, which of course represent the rotation shifts uncorrected for inclination of the Sun's axis or reduction to the edge, may be taken to indicate both constancy in

the instrumental conditions and satisfactory precision in the settings.* In view of the probable importance of level as a factor influencing solar displacements, too much significance should not be attached to the means given in Table II,† but they do at least serve as an indication that the order of magnitude of the shift at the true limb is for these lines +0·006 A. Although the danger of deriving detailed conclusions from so few as eight separate results is realised, the internal agreement between the measures shows that this value cannot be much in error. The mean for all lines of the equatorial rotation velocity is 2·07 ± ·04 km./sec., which is within 3 per cent. of the value found by Storey on applying his reduction method to these plates.

The advantages of the method described above are twofold. Firstly, the shift for each line is obtained directly in fractions of an angstrom, and not in terms of revolutions of the screw of the measuring micrometer. In this way, variations in the measured value of the shift due to slightly different

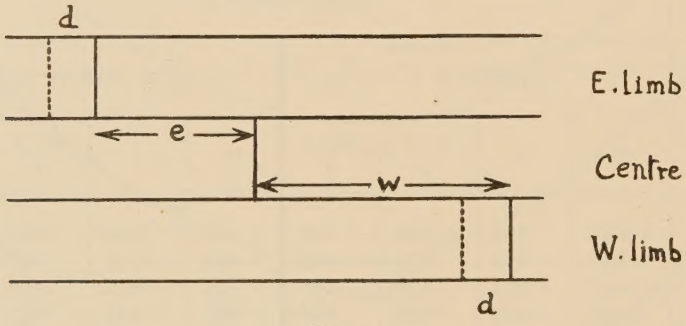

Fig. 2.

inclinations of the spectrum to the axis of the screw are automatically compensated without having to calculate a separate dispersion factor for each position of the plate on the micrometer stage. Secondly, *all* the readings for the telluric lines are used for the calculation of each shift. This reduces to a minimum the reliance placed on either the wave-length or the scale reading of any particular telluric standard. The importance of this latter condition has been shown by auxiliary measurements of telluric wave-lengths made at the Imperial College by two observers: the Rowland Revision standards differ from those calculable from the Edinburgh plates by amounts varying from +0·003 to −0·003 A.

Magnitude of the Limb Shift.—The whole history of the limb displacement is one of conflicting data and still more conflicting conclusions drawn from these data. Considering first, in chronological order, the values obtained for lines in this region by previous observers, Halm, the discoverer

* The range of spectrum measured is so small that the dependence of the rotation shift on wave-length is not detectable.

† No direct measurements of the level of origin of these lines appear to have been made, but the excitation potentials of all but the two most refrangible are quite close to one another, these and the rotation velocities indicating a level of the order of 350 km.

of the effect, found for the lines λλ 6301, 6302 a mean "limb *minus* centre" shift of +0·012 A.*

In a more comprehensive investigation, Adams † obtained a value of +0·0116 A. for the mean of 27 iron lines in the neighbourhood of the α group of atmospheric oxygen. For three lines in common with the present investigation he gives as the mean of 11 measures :

λλ 6301·517 (7)	+0·012 A.
02·508 (5)	·012
18·036 (6)	·014

Somewhat later, Evershed and Royds,‡ in investigating the variation of a few selected wave-lengths across the Sun's disc, found that the lines λλ 6280, 6301, 6302 gave a mean limb shift of +0·009 A.

In their report of an investigation of the constancy of solar wave-lengths,§ St. John and Babcock state that "from an extended series the mean limb *minus* centre shift for solar lines in this spectral region [λλ 6280–6303] is 0·007 A."

For 19 iron lines of mean wave-length 6311 A. and mean solar intensity (3), St. John ‖ found for the centre *minus* arc shift a value 0·0037 A. less than that predicted by relativity. Since at the limb the shift was 0·0015 A. greater than the Einstein prediction, it follows that for these lines there was a limb *minus* centre shift of +0·0052 A. Actually a value somewhat smaller than this might be expected, since St. John adduces evidence that lines arising at a level of 350 km., which is probably about the level of origin of the lines given in Table II, exhibit at the limb the relativity shift exactly.

No further work on these lines appears to have been published until now.

There seems to be substantial evidence here of a long-period fluctuation in the value of the limb displacement for these lines. The full curve of fig. 3 shows the tendency plainly. Although it has not in all cases been possible to find exactly when the various experiments were made, or to correct to the true limb, the variation with epoch seems unmistakable. Other groups of lines have unfortunately not been studied so completely, and no corresponding inferences can be drawn from the meagre data available. It is interesting to compare the full curve of fig. 3 with the dotted curve showing the variations recently noticed in the equatorial rotation velocity of the Sun (see *Trans. I.A.U.*, vol. iv, 1932, p. 43). The two curves agree almost exactly in period and in phase, but the amplitude of variation of the limb shift is nearly double that of the rotation velocity.

Causes of the Limb Shift.—Turning now to the explanations put forward to account for the effect, we find that the early observations of Halm,¶ Buisson and Fabry,** and Adams †† were taken as evidence that it was due to the greater effective pressure at the limb, the photospheric light having to pass through more high-pressure gas at the limb than at the centre. More

* Halm, *Astr. Nach.*, **173**, 273, 1907. † Adams, *Ap. J.*, **31**, 30, 1910.
‡ Evershed and Royds, *Kodaikanal Obs. Bull.*, No. 49, 1916.
§ St. John and Babcock, *Ap. J.*, **55**, 36, 1922.
‖ St. John, *Ap. J.*, **67**, 195, 1928. ¶ Halm, *Astr. Nach.*, **173**, 273, 1907.
** Buisson and Fabry, *Comptes Rendus*, **148**, 1741, 1909 ; *Ap. J.*, **31**, 116, 1910.
†† Adams, *Ap. J.*, **31**, 30, 1910.

recent deductions, however, either from direct spectroscopic observations * or from theoretical considerations,† have shown clearly that the pressure in the reversing layer is extremely low.

Evershed and Royds ‡ have put forward the "startling and perhaps incredible supposition" of a repulsion of the solar gases by the Earth—an influence which is, however, not shared by the other planets. This effect would presumably give a red shift (compared with terrestrial sources) increasing progressively towards the limb, where there would be little to prevent the gases from reaching a high velocity. This conclusion, though

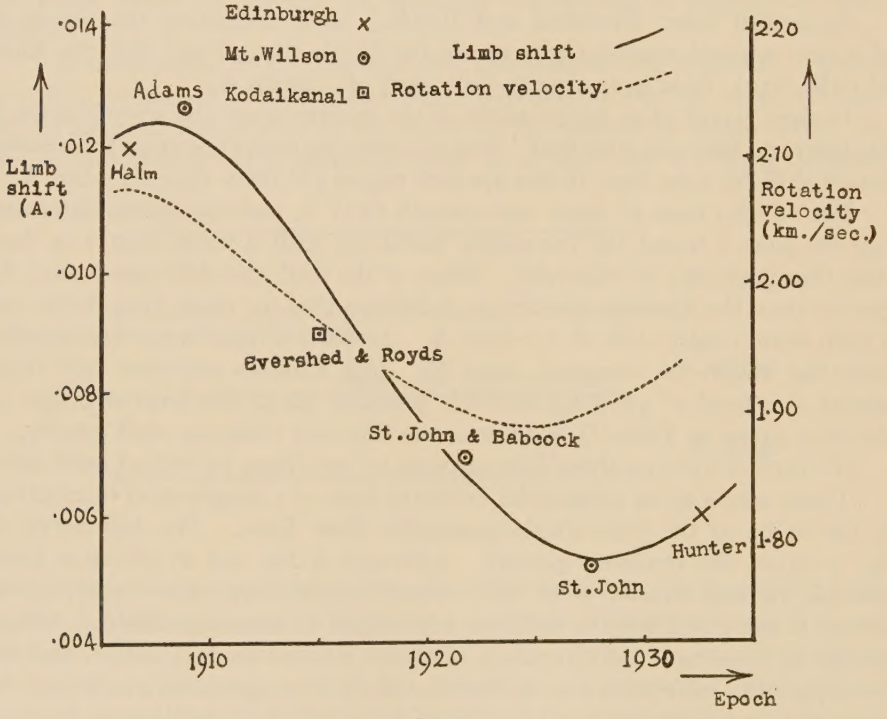

FIG. 3.

supported by Evershed's observations § on the light from the hidden hemisphere of the Sun reflected from Venus, has not been confirmed by similar measurements made by St. John, || who correlates his results with altitude and attributes them to atmospheric refraction.

The theory that anomalous refraction will produce displacements of the right order of magnitude ¶ has never received much support from practical astrophysicists,** and the weight of observational evidence is against it.

* St. John and Babcock, *Ap. J.*, 60, 32, 1924.
† Fowler and Milne, *M.N.*, 83, 419, 1923.
‡ Evershed and Royds, *Kodaikanal Obs. Bull.*, Nos. 39 (1914), 49 (1916).
§ Evershed, *Observatory*, 42, 51, 1919. || St. John, *Ap. J.*, 53, 380, 1921.
¶ *E.g.* Julius, *Ap. J.*, 31, 419, 1910 ; Albrecht, *Ap. J.*, 41, 333, 1915 ; 44, 1, 1916.
** St. John, *Ap. J.*, 41, 28, 1915 ; 44, 311, 1916 : Royds, *Kodaikanal Obs. Bull.*, No. 48, 1915.

In his "Evidence for the gravitational displacement of solar lines," * St. John attributed the uniform red shift of the Fraunhofer lines relative to terrestrial sources to the Einstein effect, and the limb shift to the effects of upward vertical currents in the Sun's atmosphere. The Doppler effect due to these reaches a maximum at the Sun's centre—where the Einstein shift is therefore greatly reduced—but vanishes at the limb, where the red shift should agree with Einstein's prediction. On this hypothesis, then, the effect is to be regarded not as a shift to the red at the limb, but rather as a shift to the violet at the centre; and its determination for any line resolves itself into the determination of the velocity of outflow of the vapour absorbing that line.

The suggestion may be tested in two ways:

(*a*) by comparing the limb *minus* arc shifts with those predicted from the equivalence hypothesis, and

(*b*) by finding experimentally the law by which the wave-length of a Fraunhofer line varies across the Sun's disc.

The first test yields conflicting evidence. At λ 3900, St. John * concludes for a "residual limb shift" of +0·0022 A. (*i.e.* a limb *minus* arc shift of 0·0022 A. in excess of Einstein's prediction), whilst Evershed's figures † are uniformly higher than this (mean for Fe lines, +0·0063 A.).

The procedure in (*b*) above was actually carried out by Evershed and Royds ‡ before St. John's interpretation was advanced. Curves were obtained for the iron lines $\lambda\lambda$ 6280, 6301, 6302, and for others in the more refrangible part of the spectrum, but naturally no comparison with calculated values was made. It is easy to show that the wave-length of a line at any point on the Sun's disc should be displaced relative to a terrestrial source by an amount corresponding to a line-of-sight velocity of

$$0{\cdot}634 + v(1 - r^2/R^2)^{\frac{1}{2}} \text{ km./sec.},$$

where v = velocity (in km./sec.) of radial streaming at the level in which the line considered is absorbed (negative if upward);

r = distance, as seen from the Earth, of the observed point from the centre of the disc; and

R = radius of the Sun.

The first term corresponds, of course, to the gravitational shift expressed as a velocity, and is constant at all points. The second term varies from v at the centre of the disc to zero at the limb.

This expression has been plotted against the ratio r/R for three convenient velocities in fig. 4 (full lines). The broken line represents the mean variation found by Evershed and Royds for the lines $\lambda\lambda$ 6301, 6302. The line λ 6280 has not been included since its very much lower excitation potential suggests an origin at a different level in the Sun's atmosphere; its inclusion would in any case make the agreement worse. For comparison purposes, a similar procedure has been carried out for the more recent results of

* St. John, *Ap. J.*, 67, 195, 1928.

† Evershed, *M.N.*, 91, 260, 1931.

‡ Evershed and Royds, *Kodaikanal Obs. Bull.*, No. 49, 1916.

Freundlich, Brunn and Brück* (dotted line). It will be seen that the observed variations cannot possibly be reconciled with those predicted by St. John. The values found in the two investigations appear to agree fairly well, the two observed curves of fig. 4 differing from the calculated ones in the same general way. It should be specially noted that, in the absence of any data bearing on the subject, the curves have been drawn assuming an exact agreement with the relativity prediction at the limb.

Burns† has also tested the radial streaming theory by measuring the

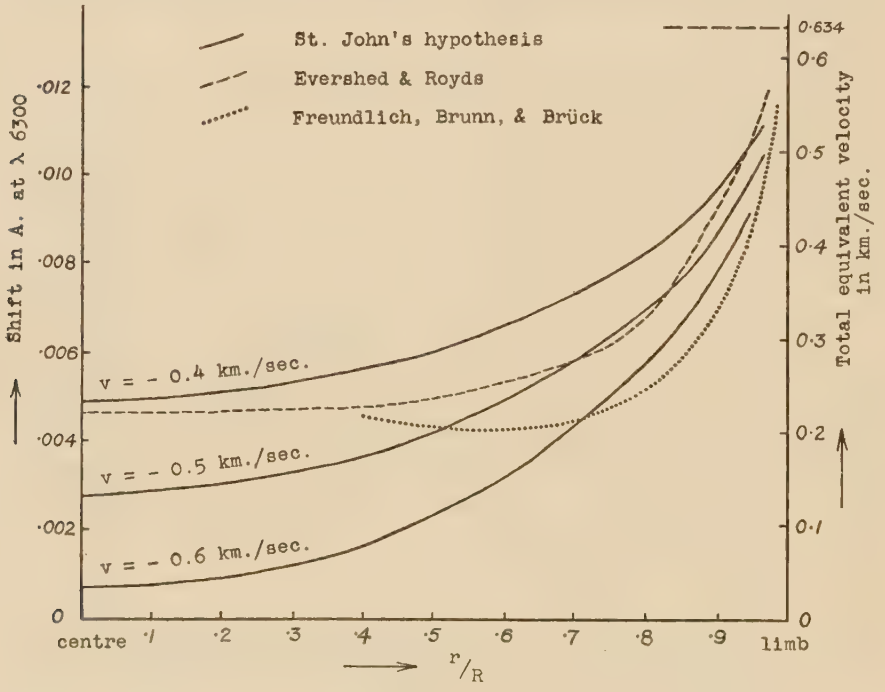

FIG. 4.

magnitude of the displacement at a position on the solar disc corresponding to $r/R = 0{\cdot}92$. At this point he found that the effect was negligible.

A recent suggestion by Evershed‡ attributes both the excess shift at the limb and the deviations from the theoretical curves to the "objectionable Earth effect" in combination with St. John's radial movements and the relativity displacement. Although this threefold mechanism may be used to account for the former difficulty, it is hard to see how it can explain the latter. For though the exact form of the function representing the velocity of recession of the Earth-repelled vapour is not at once apparent, it must surely increase continuously on passing from the centre to the limb. The theory would therefore involve an increase of slope at all points on the

* Freundlich, Brunn and Brück, *Zeits. f. Astrophysik*, **1**, 43, 1930.
† Burns, *J.O.S.A.*, **20**, 220, 1930.
‡ Evershed, *M.N.*, **91**, 260, 1931.

theoretical curves of fig. 4. But it is obvious that for more than three-quarters of the way across the disc, the curves are already too steep for reconciliation with observation. In order to make the curves agree, the recessive velocity would have to be large at the centre and limb, and small or even negative at some intermediate point.

It seems clear that no hypothesis yet put forward will account for all the facts. Until one is forthcoming, the large deviations from the predictions of Einstein will give reason for doubting even the assignment of the major part of the red displacements to the gravitational effect. Any future explanation retaining the relativity shift as valid must account for the excess shift at the limb and the defect at the centre, together with the variations in these quantities with line intensity and with epoch, and also for the observed form of the function representing the change in wave-length of a Fraunhofer line across the solar disc.

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THE CONCEPT OF TIME.

R. A. Sampson, F.R.S.

The purpose of the following paper is to provide a logical basis for experience in relation to the mutual influences of two particles of matter.

It is also offered as an example showing that certain features which are often regarded as essential, *e.g.* a curvature of space-time, can be dispensed with.

It is confined to what is called "classical dynamics,"—that is to say, the particles are supposed to be particles of matter, and it does not refer to the quantum conditions, nor to non-commutative quantities, nor to the electromagnetic equations.

It consists for the most part of rearrangements of familiar notions.

§ 1. *Initial*

Consider a function, related to two particles, or related incidents, which may be called O, O',—

$$H \equiv E + E'$$
$$\equiv c(P_1 p_1{}^2 + P_2 p_2{}^2 + P_3 p_3{}^2 + \Pi\pi^2)^{\frac{1}{2}} + c(P_1' p_1'{}^2 + \ldots + \ldots + \Pi'\pi'^2)^{\frac{1}{2}}, \quad (1)$$

where

c is a constant,

p_1, p_2, p_3, π are referred to as the "momenta" of the first particle O,

p_1', p_2', p_3', π' are referred to as the "momenta" of the second particle O';

and

$P_1, \ldots \Pi, P_1', \ldots \Pi'$ are not functions of the momenta but are presumed to be functions of certain other quantities $q_1, \ldots$, called the "co-ordinates."

The invariance of this form for different choice of axes will be demonstrated in § 3, where it is shown to be a constant.

§ 2

From this definition it follows that H is a homogeneous function of the first degree in the momenta, and hence

$$p_1\frac{\partial H}{\partial p_1} + \ldots + p_1'\frac{\partial H}{\partial p_1'} + \ldots = H.$$

Introduce differential quantities dq, defined by

$$\frac{\partial H}{\partial p_1}d\tau = dq_1, \ \ldots \ \frac{\partial H}{\partial \pi_1}d\tau = cdt. \tag{2}$$

The quantities q, or ct, are the quantities referred to as co-ordinates in § 1. As to the definition of the differential values dq, see § 21. As to the integral values q, see § 7.

By the definition (2) we also introduce a quantity $d\tau$. $d\tau$ is unobservable and is merely a mathematical token; it is supposed to be the necessary, common measure of the related changes which go on between the incidents. τ is of the character of a co-ordinate. It is supposed to be absent from the expression for H.

Some remarks will be made later, in § 21, as to the absolute character of τ.

If a name is wanted, τ may be called the "absolute time," but I shall not need a name.

Any quantity which the operation $\frac{d}{d\tau}$ reduces to zero is called a constant.

From the definition (2) it follows:

$$p_1dq_1 + \ldots + \ldots + \pi cdt + p_1'dq_1' + \ldots + \ldots + \pi' cdt' - Hd\tau = 0. \tag{3}$$

§ 3

Consider the two incidents O, O' as related to one another. Each incident may be deemed, from its own point of view, a singularity. If there is to be any common discussion, a common law governing O and O' must be assigned. Denote by the variational symbol δ the independence of the two incidents. This will not require any difference from the usual practice.

Assign as the law

$$\delta d - d\delta = 0.$$

From the common character of $d\tau$, we have $\delta d\tau = 0$.

Operate upon the expression (3) with δ :

$$\begin{aligned} 0 &= \delta(p_1 dq_1 + \ldots + p_1' dq_1' + \ldots - H d\tau) \\ &= d(p_1 \delta q_1 + \ldots + p_1' \delta q_1' + \ldots) \\ &\quad - dp_1 \delta q_1 - \ldots - dp_1' \delta q_1' - \ldots \\ &\quad + \delta p_1 dq_1 + \ldots + \delta p_1' dq_1' + \ldots - \delta H d\tau. \end{aligned}$$

We also have

$$0 = \frac{\partial H}{\partial p_1}\delta p_1 + \frac{\partial H}{\partial q_1}\delta q_1 + \ldots - \delta H.$$

Hence from the definition (2)

$$0 = \delta p_1 dq_1 + \frac{\partial H}{\partial q_1}\delta q_1 d\tau + \ldots - \delta H d\tau.$$

Eliminate δH by this equation ; we have

$$\begin{aligned} 0 &= d(p_1 \delta q_1 + \ldots + p_1' \delta q_1' + \ldots) \\ &\quad - \left(\frac{\partial H}{\partial q_1} d\tau + dp_1\right)\delta q_1 - \ldots \end{aligned} \tag{4}$$

The co-ordinates are supposed to be independent, until they are restricted by adoptions. See § 10. If they are independent, this relation implies

$$-\frac{\partial H}{\partial q_1} d\tau = dp_1, \tag{4a}$$

and

$$\left[p_1 \delta q_1 + \ldots + p_1' \delta q_1' + \ldots\right]_A^B = 0, \tag{4b}$$

where B, A denote certain terminal points.

If the co-ordinates are not independent, less is implied. The modifications will be treated when the adoptions are made.

Note that (4a) is the usual correlative of (2) ; (4b) is merely a terminal version of these. Its form will also be treated when adoptions are made.

It also follows :

$$\begin{aligned} dH &= \frac{\partial H}{\partial \tau} d\tau + \frac{\partial H}{\partial p_1} dp_1 + \frac{\partial H}{\partial q_1} dq_1 + \ldots \\ &= \frac{\partial H}{\partial \tau} d\tau \\ &= 0, \end{aligned}$$

since τ has been supposed absent from H. Hence H is a constant.

As to the constituents of H, viz. E and E', see § 16 (30).

§ 4

From the definition of dq,

$$\frac{E}{c^2}\frac{1}{P_1} dq_1 = p_1 d\tau, \ldots$$

Also from the definition of E,

$$E^2 d\tau^2 = c^2(P_1 p_1{}^2 + \ldots)d\tau^2.$$

Substitute for $p_1, \ldots$, and multiply by c^2/E^2;

$$c^2 d\tau^2 = dq_1{}^2/P_1 + \ldots + (cdt)^2/\Pi, \tag{5}$$

and similarly

$$= dq_1{}'^2/P_1' + \ldots + (cdt')^2/\Pi'.$$

We may compare this with the postulated invariant of General Relativity:

$$d\sigma^2 = g_{\mu\nu} dx_\mu dx_\nu.$$

Note, however, that this postulate implies a relation between the differential geometries of O and O'; or alternatively the element $d\sigma^2$ or $c^2 d\tau^2$ is invariant because it is compared with itself.

§ 5. *Absent Co-ordinates*

By the equations (2), $d\tau$ may be eliminated in favour of dq, the change in any co-ordinate.

This step is most usually profitable when the corresponding co-ordinate is absent from H. In such a case the co-ordinate may serve in every respect as a substitute for τ.

Thus in the rotation of the Earth, the angle of rotation is not present. Hence the phase of rotation may be used as a measure of time.

In the case of the motion of the Earth in its orbit, the angle of revolution is again not present explicitly. Eliminate the time and we get an equation between the co-ordinates, which is the equation of the orbit.

We are reminded of Newton's "*durationis per motum mensura.*"

On closer examination, it will be seen that neither of these quantities is really absent. The former declares itself through tidal friction, *i.e.* through the position of the Moon. The latter appears through perturbations by other planets, and declares itself through the position of those planets. But none of these positions are "absolute," and we have the following characteristic situation. We assert that the Earth rotates, and that it revolves round the Sun, though these statements imply a direction which does not move, which it is impossible to assign. Two directions will suffice for all the phenomena seen from the Earth, and these are very numerous. There is then only a demand for two fixed directions in order to make all the statements "absolute"; but we are unable to assign them.

If, besides the positional co-ordinates, we include t as relative, the necessary number of data is three, not two. The progress of t may be fixed, as we believe, by the decay of radium. But we are unable to render any datum absolute. We are in the like position.

The difficulty is much more striking when we deal only with two particles or incidents, whose co-ordinates are little more numerous than the unassignable absolutes.

When a co-ordinate is absent, by 4 (*a*) the corresponding momentum is constant.

Absence of any co-ordinate can only occur if we forego complete generality, since it implies the choice of appropriate axes of reference.

But it may become important if there are only two particles to consider, O and O'. In that case, experience shows that O and O' modify one another's motion merely owing to their relative position in space-time—whatever that may mean. That is to say, the axes can be chosen so that all except one of the co-ordinates are absent.

In the more common case when H does not involve the local times, it means that we can choose lines along which t and t' change, which do not enter in the expression for H.

In spite of the restriction that H is independent of the local times, it will be seen in § 6 that the latter are affected by the gravitational field and the motion.

§ 6. *Mass and Local Time*

As a rule t, t' will be absent from the expression for H; so that without practical loss, we may restrict the theory to their absence.

Following § 5, we have then

$$\pi = const., \qquad \pi' = const.$$

Thus two parameters are introduced, necessarily.

But in the case where the momenta, other than π, are unimportant, and also the gravitational field, represented by the difference of Π from unity, the definition of E gives

$$E = c\pi.$$

Compare with the familiar equation

$$E = mc^2, \tag{6}$$

where c is now taken, as henceforward, for the velocity of light, then it is clear that it will accord with practice if the parameter introduced is identified with m, the rest mass, or

$$\pi = mc. \tag{7}$$

Choosing the symbol m so, we find from the definition of E, the generalization

$$E = mc^2\Pi^{\frac{1}{2}}\left[1 - \frac{1}{c^2P_1}\left(\frac{dq_1}{d\tau}\right)^2 - \ldots\right]^{-\frac{1}{2}};$$

$\frac{dq_1}{d\tau}$, . . . are unobservable; it will generally suffice if we write

$$E = mc^2\Pi^{\frac{1}{2}}\Big/\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \tag{8}$$

where v represents the observed velocity.

Some further remarks upon this form will be made in § 20.

If we eliminate τ, using these values of π, π' we get from (2)

$$m\frac{dq_1}{dt}=\frac{P_1}{\Pi}p_1$$

$$m'\frac{dq_1'}{dt'}=\frac{P_1'}{\Pi'}p_1',$$

where, it is to be noted, that generally $t\neq t'$; more definitely

$$\frac{E}{mc^2}\cdot\frac{dt}{\Pi}=\frac{E'}{m'c^2}\cdot\frac{dt'}{\Pi'}=d\tau.$$

We shall make a choice below so that $P_1=\Pi$; these equations read, then,

$$p_1=m\frac{dq_1}{dt},\qquad p_1'=m'\frac{dq_1'}{dt'},\tag{9}$$

or "momentum equals rest mass × local velocity"—with parallel statements for the angular co-ordinates.

Notice that m is thus introduced into expressions for the momenta.

The parameter m is the link that binds time with the other co-ordinates, or say, it draws the line of separation between geometry and mechanics.

The foregoing choice of m represents an adoption of a scale for the ratios

$$dq_1 : dq_2 : dq_3 : cdt$$

if H, as originally defined, is to fall in with existing practice.

Motion, then, as illustrated by $\frac{dq}{dt}$, is presented as an instantaneous scale between the elements of local co-ordinates. We may speak of motion without feeling that there is any mystery in it.

Usually relations between q_1, q_2, q_3 form a "geometry" ; but we may, without prejudice, treat any three of the four co-ordinates as forming a geometry ; but it will be convenient to follow the usual custom, owing to the very large value of the constant c, which, whether it is absolute or not, forms a natural cleft separating problems of geometry and mechanics.

Some further geometrical considerations will be found in the next section.

§ 7. *Nature of Co-ordinates*

This is the proper place to say that any space-time contemplated is flat, or Euclidean.

This is chosen partly for contrast, to show that all the results can be calculated without curvature of space-time.

But partly too it is dictated by convenience, because the language is familiar and it is convenient to be able to talk of a straight line or a direction without having to question whether such things exist ; partly too because the peculiarities of flat space are well known, and I find it difficult to avoid error when, *e.g.*, arguing away such things as the equatorial bulge of the Earth.

Frequently, however, the statements apply without change to any space-time.

In § 2, dq only has been introduced. But in § 1, the P must be supposed functions of the integral q. The last must be viewed as the sum $\int dq$, taken between appropriate limits.

The summation, $\int dq$, is merely arithmetical. The difficulty arises in the ascertainment of dq. Some remarks are given in § 21, showing how dq is imagined to be apprehended. This view supposes, for its application, that the observer O occupies all the successive positions up to O', in order to ascertain the values of dq. In general this will not be the case. We have to face the impossibility of assigning what is at another point of space-time, that is to say, neither at O nor at O', where alone observations can be made. The only course is an adoption, *i.e.* to adopt a "messenger" at an agreed rate of progress. The usually adopted messenger is a pulse of light. Strictly it is a convention. It may, none the less, be useful.

An instance of the utility of adoptions is the rule that is necessary to adopt in order to define completely the product of two fractions.

Here we take, as usual, that a pulse of light is found to move with a constant velocity c, no matter what are the local velocities of O or O'.

§ 8. *Special Case*

As this matter is fundamental, it may be well to treat it in its simplest form, that is to say, with the smallest number of co-ordinates and in the absence of a gravitational field.

This method permits immediate comparison with Special Relativity.

In equations (3), (5), take then $dq_2 = dq_3 = 0$, $P = \Pi = 1$ and ignore the second particle; the equations reduce to

$$p_1 dq_1 + \pi c dt - E d\tau = 0, \tag{3'}$$

where p_1, π are constant; and

$$c^2 d\tau^2 = dq_1^2 + c^2 dt^2. \tag{5'}$$

If the "body" is a pulse of light, it will appear from § 19 that we must take $p_1 = \pi$; to preserve the parallel, write the equation (3') in the form

$$c dt - dx = const. \tag{3''}$$

In this form it is the well-known allowance for the transmission of light from a distance, reading that, for a shift of O in space-time, the shift in space must be set against the added time.

The same may be treated with more generality nearly as Einstein did originally. Let O, O' be two observers; let u be the velocity which O ascribes to O', and $-u$ that which O' ascribes to O. Assume that O and O' are unambiguous in denoting any third incident. This means that a bilinear substitution is all that is requisite. Consequently we may write the transformation from $O(x, t)$ to $O'(x', t')$ in the form

$$\lambda dx = dx' + u dt',$$

$$\lambda dt = \frac{dx'}{w} + dt', \tag{10}$$

where we write $uw = c^2$; these imply

$$\lambda' dx' = dx - udt,$$

$$\lambda' dt' = -\frac{dx}{w} + dt,$$

provided

$$\lambda\lambda' = 1 - u/w = 1 - u^2/c^2.$$

Then the velocities, $v \equiv \frac{dx}{dt}$, $v' \equiv \frac{dx'}{dt'}$, ascribed by O, O' to any, the same, incident are related by the bilinear equation

$$\frac{vv'}{w} + v - v' - u = 0. \tag{11}$$

The two common solutions of this equation are $v = v' = c$, and $v = v' = -c$. Other forms of the relation are

$$\left(1 - \frac{u}{v}\right)\left(1 + \frac{u}{v'}\right) = 1 - \frac{u}{w} = \left(1 - \frac{v}{w}\right)\left(1 + \frac{v'}{w}\right); \tag{11a}$$

$$v = \frac{v' + u}{1 + v'/w}, \qquad v' = \frac{v - u}{1 - v/w}; \tag{11b}$$

$$\frac{c - v}{c + v} = \frac{c - u}{c + u} \cdot \frac{c - v'}{c + v'}. \tag{11c}$$

We remark that there are only two constants in this bilinear equation, as compared with three that are required if it is to be quite general. The third has been embodied in the condition that O and O' attribute a common value to the velocity c.

If λ, λ' are any solution of (10), it follows that $\kappa\lambda$, λ'/κ are also a solution. Hence all solutions may be derived from any one.

That means that any two solutions differ only in their scale, which is a matter for adoption. Hence if any solution explains a phenomenon, as, *e.g.*, the Doppler effect, any other will do the same.

Take then

$$\lambda = \lambda' = (1 - u^2/c^2)^{\frac{1}{2}}. \tag{12}$$

We have Lorentz's equations, which Einstein also arrived at. He chose equality deliberately so as to make recession identical with approach. I do not see that this is necessary or legitimate. In so doing we lose equation (3″), for the invariant now runs

$$c^2 dt^2 - dx^2 = const.; \tag{13}$$

and also a certain paradox is introduced, exposed by Einstein in the following words :— *

"Suppose there are two synchronous clocks at A, and one of them is moved in a closed curve with constant velocity till it returns to A again,

* "Zur Elektrodynamik bewegter Körper," *Ann. d. Physik*, **17** (1905), § 4.

occupying, say, t seconds, then the latter clock on arriving at A will be $\frac{1}{2}t(u/c)^2$ slow on the one which remained unmoved. We conclude that a balance clock, placed at the equator, must run slow, by a very small amount, upon an exactly similarly constructed and similarly treated clock placed at the pole."

Since the standard clock for any point upon the Earth is a gyroscope, namely, the Earth's own body, the statement runs that the Earth in its rotation goes slow upon itself and does so cumulatively.

A contradiction in terms could hardly be more complete. In spite of this, this paradox and others have been widely exploited since, though a paradox is quite foreign to the spirit of mathematics.

If we set

$$\kappa = \sqrt{\left(\frac{c+u}{c-u}\right)},$$

we have

$$\lambda\kappa = 1 + \frac{u}{c}, \qquad \frac{\lambda}{\kappa} = 1 - \frac{u}{c}; \tag{14}$$

the invariant (3″) is preserved, and no paradox presents itself, as is evident. The "time" gained on the outward journey is lost again on the return.

For this reason I do not adopt the substitution $dt = idt_1$, etc., leading to a version of (5′),

$$c^2 d\tau^2 = c^2 dt_1^2 - dq^2 ;$$

I regard this step as made merely to preserve an analogy, which is misleading, in order to stress the likeness to Lorentz's invariant.

Notice that there is nothing in the foregoing that requires no velocity to exceed c. The exceptional character of the transmission of light is marked only by identifying $\pm c$ with the double values of the bilinear relation.

§ 9. *Geometrical*

The following treatment assumes that a geometrical representation is adequate for the quantities described.

More, it assumes that a geometry exists ; as to this, see § 21.

Let O, O' represent the two incidents. To represent the domain of O, set two axes at right angles, Ωx, Ωct, and do the same for O'.

Then the fact necessary to represent is that a gradient at O, viz. $dx/d(ct)$, uniquely determines one at O', and conversely.

We have the following elements to contemplate :—

(1) The co-ordinates of O with respect to the chosen axes.

(2) The set of the axes relative to the gradient at O with the like quantities for O'.

Of these,

(1) does not enter at all ;

(2) is at choice, but for any axes, if we draw the lines $\Omega\Psi'$, $\Omega\,\Phi$ representing $dx/d(ct) = \pm 1$, the gradient at O must lie within these directions, if the velocity is less than c.

Connect diagram (I) with diagram (II).

This may be done by setting $\Omega' x'$ along Ωx, at an assigned distance $\Omega\Omega'$.

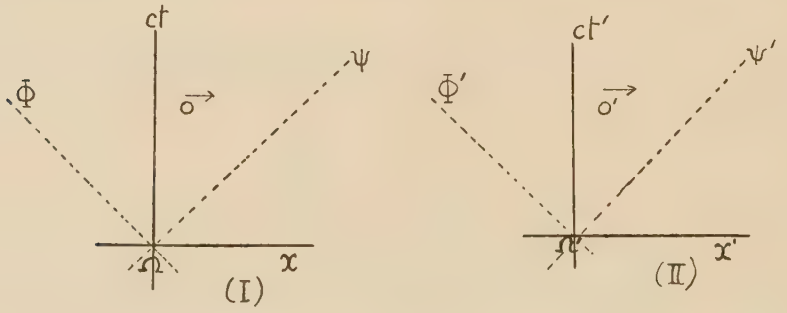
The relative gradient which O attributes to O' will then not depend upon the axes at all, and will be equal and opposite to that which O' attributes to O.

Note that this allows for difference of the co-ordinates (x, t) from (x', t').

For any other direction, apart from the given gradients, this will not be true ; but as no observation can be made, it cannot be challenged.

The statement that O attributes to O' a gradient equal and opposite to that which O' attributes to O, may be made to appear an assumption. A more general case would say that O attributed to O' a velocity or gradient which was different to that attributed by O' to O. In this case half the sum of the gradients might be treated as not entering at all, being common to both attributions, while half the difference would fall under the discussion above, being equal and opposite for O and O'.

The question is not an idle one, because the common velocity or gradient would appear to add to the "double velocities" of the (1, 1) system, $\pm c$.



This would account for the Doppler displacement, upon pre-relativity lines, but would not serve to explain the Michelson-Morley experiment.

But the statement cannot stand ; it assumes that O and O' have knowledge of some third body with respect to which they estimate the common velocity. But there is no such body. Indeed "equal" or "equal and opposite" is the extent of their knowledge.

The foregoing deals with the case when the particles are unrelated causally. But experience shows that two particles, O, O', change one another's motion. It is assumed that they do not do so discontinuously. Hence the cause of change must operate through the gradients of these gradients as scales. Apply the same diagrams to these cases, except that there is no reason now to attach any significance to $\Omega\Psi$, $\Omega'\Psi'$. This means that the action may reach any value.

We then have, as before, that the measures of the effects are equal and opposite, or "action and reaction are equal and opposite."

In the gravitational case, where the changes depend upon the positions of O and O', the relative position acquires a significance. If we figure OO' as a straight line, its direction becomes the datum, and the only datum, for the action, giving it a definite direction, because the perpendicular direction is indefinite.

If there is no significance in the directions $\Omega\Psi$, $\Omega'\Psi'$, the "double values" of the (1, 1) relation become infinite, and we have that O attributes to any third incident a certain acceleration which differs from that attributed by O' by a certain amount; that is to say, we have Newton's conception *plus* Maxwell's addition, which corrected Newton's proof of the null effect of two bodies on one another, by pointing out that an acceleration of the combined body would be unobservable.

Remark that nothing in the above requires constancy in any quantity.

§ 10. *Co-ordinates. General Case*

We have, to start with, co-ordinates q_1, q_2, q_3, ct, q_1', q_2', q_3', ct', and also τ; that is to say, two 4-manifolds and a connecting bond.

Recall that the geometry of each 4-manifold is Euclidean.

Where there is no information, there is liberty of choice.

This liberty may be used to secure simplification.

Any adoption must (1) accord with experience, (2) be logically permissible, and (3) should be exposed as far as possible.

Nothing is observable except at O and O'.

While the general system is complete and universal, it would be idle to make it apply to places where no observation can be made.

The adoptions, then, will require to be applicable to O and O', but otherwise I shall aim only at making them simple.

Any adoption may have consequences in the equations (4).

The aim will be to adapt the system towards polar co-ordinates, as far as permissible.

Begin with q_3, q_3'. Consider the sections by the planes $t = const.$, $t' = const.$ of each 4-manifold.

The sections will be ordinary geometry of 3 dimensions.

q_3, q_3' may be treated as azimuthal angles about an axis, which may be considered OO'.

q_3 will not in general be present in H. Hence $p_3 = const.$ Similarly $p_3' = const.$ These constants may both be made zero by choosing the planes q_3, q_3' at right angles to the relative velocities of O, O', which, as we have seen in the last section, are parallel and opposite. Therefore the particles continue to move in one plane. Azimuthal angle is not significant. Therefore I shall drop q_3, p_3, q_3', p_3' from future reference.

We have, then, three co-ordinates only for O, viz. q_1, q_2, ct, and the like for O', together with a bond τ. We may figure the relations for O by a three-dimensional geometry which is connected with that for O' by such equations as (2). The two spaces may be considered overlapping, or identical. It must be held that O knows so much of O' as to attribute a distance to O'. As to this statement, see § 21. Let this distance be r. Then taking the suggestion from the vector triangle OPO', viz. $\overline{OP} + \overline{PO'} + \overline{O'O} = 0$, adopt as the relation

$$q_1 e^{iq_2} + q_1' e^{iq_2'} = r.$$

I shall frequently write hereafter, as usual, $r = 1/u$.

As to the permissibility of this adoption, the question of its logic may be referred to the discussion of the logic of the vector equation of a triangle. As an equation, q_1' denoting O', the line OO' will be given by $q_2=0$, $q_1=r$. Similarly $q_1=0$ denotes o, and $q_2'=0$, $q_1'=r$. Therefore the equation implies that the distance which O attributes to O' is the same as that which O' attributes to O. We may denote the "straight line" OO' by $q_2=q_2'=0$. At O or O' when $q_1=0$ or $q_1'=0$, the values are indeterminate; so that we may take them at what we like. Hence as far as observable facts are concerned, we may adopt

$$\delta q_2=\delta q_2' \tag{15a}$$

$$\delta q_1+\delta q_1'=0. \tag{15b}$$

Inserted in equations (4), these imply

$$dp_2+dp_2'=-\left(\frac{\partial H}{\partial q_2}+\frac{\partial H}{\partial q_2'}\right)d\tau$$

and

$$dp_1-dp_1'=-\left(\frac{\partial H}{\partial q_1}-\frac{\partial H}{\partial q_1'}\right)d\tau.$$

In general q_2, q_2' will not appear in H; hence

$$p_2+p_2'=const.=k, \text{ say} \tag{16}$$

where p_2, p_2' are not disconnected momenta but are those related with the foregoing adoptions.

This is the familiar equation expressing the "conservation of angular momentum."

Notice that it applies to the general case, and is true for any permissible values of the other co-ordinates, t, t'.

Notice too that k cannot be made zero. It depends upon the relative velocity which O attributes to O', other than in the line OO'.

§ 11

Deferring the further consideration of these equations, consider the expression for H. The factors Π, P_1, P_2, Π', P_1', P_2' in (1) are intended to provide for the gravitational field. I write

$$\Pi=1-\frac{\mu u}{c^2}\ldots \quad \Pi'=1-\frac{\mu' u}{c^2}\ldots \tag{17}$$

where μ, μ' are two parameters, representing the first coefficients of a power series of a common quantity n.

Note that the gravitational field of O is "produced" by O', so that $\mu\to 0$ if $m'\to 0$, and conversely.

P_1, . . . have analogous developments with which we are not now concerned.

The assumption made in (17) is no more than that the effect of O or O' at great distances, $(u\to 0)$, is insensible. The same assumption was made in

General Relativity as first put out; it is demonstrable that it leads there to an untenable position (*cf.* Einstein, *Theorie de la Relativité*, Hermann et Cie, 1933, p. 105); consequently the theory is corrected by a small term expressing positive powers of r. This leads to an "expanding universe." Opinion is not at one on this point, nor whether experience calls for it. I do not refer to it further.

Using these developments, we have

$$E = c\left\{\left(1 - \frac{\mu u}{c^2}\right)m^2c^2 + p_1^2 + \ldots\right\}^{\frac{1}{2}}$$

$$= mc^2 - \tfrac{1}{2}m\mu u + \frac{1}{2m}p_1^2 + \ldots$$

and

$$H = E + E' = (m+m')c^2 - \tfrac{1}{2}(m\mu + m'\mu')u + \frac{1}{2m}(p_1^2 + \ldots)$$

$$+ \frac{1}{2m'}(p_1'^2 + \ldots) + \text{terms of order } c^{-2} \ldots \quad (18)$$

In more complicated cases, it is evident that we may write

$$H = const. + V + T,$$

where V is the potential energy, T the kinetic energy expressed in terms of the momenta; in other words, our system leads to the ordinary Hamiltonian equations. The geometrical equations between the co-ordinates reduces number of variables to the number of "degrees of freedom" possessed by the bodies.

Since the Hamiltonian equations are known to express all verifiable cases in classical dynamics, apart from such as are influenced by the finite value of c, we may conclude that the form for H postulated in § 1 is justified.

Further, it is only a question of choosing the cases to which we apply the equations, in order to deduce from them Newton's Laws of Motion, and equally any equations of motion that are based upon the latter.

As a postulate, the form (1) for H is certainly not self-evident. If a statement is self-evident, it is either tautological, or derived from metaphysics. The justification of form (1) is experience. The best we can do with experience is to render its statement compact.

§ 12. *Solution when c^{-2} is Neglected*

This proceeds upon familiar lines, so that no meticulous care is demanded. None the less the equations have a certain interest. By § 6 we see that in this case

$$dt = dt'.$$

The function H is, by § 11,

$$H = const. - \tfrac{1}{2}(m\mu + m'\mu')\frac{1}{r} + \frac{1}{2m}(p_1^2 + p_2^2 q_1^{-2}) + \frac{1}{2m'}(p_1'^2 + p_2'^2 q_1'^{-2}) \quad (19)$$

where $$q_1 + q_1' = r = 1/u.$$

Place the origin at the point,—the "centroid"—where

$$\frac{p_2}{q_1}=\frac{p_2'}{q_1'}=\frac{k}{r}; \tag{20a}$$

since $$dq_2=dq_2',$$

this requires $$mq_1=m'q_2'. \tag{20b}$$

By § 3, an integral is $H=const.$ In virtue of these relations, this leads to

$$\frac{mm'}{m+m'}\left(\frac{dr}{dt}\right)^2+\frac{m+m'}{mm'}\frac{k^2}{r^2}-(m\mu+m'\mu')\frac{1}{r}+\gamma=0, \tag{21}$$

where γ is a constant; or

$$\frac{mm'}{m+m'}r^2\left(\frac{dr}{dt}\right)^2=\gamma(\alpha-r)(r-\beta),$$

if

$$\gamma(\alpha+\beta)=m\mu+m'\mu', \qquad \gamma\alpha\beta=\frac{m+m'}{mm'}k^2.$$

Suppose α, β are real; then r oscillates between α and β. To integrate, put

$$r=\alpha\sin^2\phi+\beta\cos^2\phi;$$

then

$$r\Big|_{\beta}^{\alpha}=\phi\Big|_{0}^{\frac{\pi}{2}},$$

and

$$(\alpha+\beta)\phi-(\alpha-\beta)\sin\phi\cos\phi=t\left(\gamma\frac{m+m'}{mm'}\right)^{\frac{1}{2}}+const. \tag{22}$$

Introduce the usual constants, T, n, a, e, so that

$$T=2\pi/n, \qquad a(1+e)=\alpha, \qquad a(1-e)=\beta;$$

then this equation gives

$$a^3n^2=(m\mu+m'\mu')\frac{m+m'}{2mm'},$$

on inserting the values (37) of μ, μ' reached in § 18, so that

$$m\mu+m'\mu'=2Cmm',$$

where C is the "constant of gravitation," it reads

$$a^3n^2=C(m+m'), \tag{23}$$

which is Kepler's Third Law in the usual form. Some remarks will be made upon it below.

By eliminating the time in the usual way from the equation $H=const.$, we get the equations to the orbit; they are written down as they will be wanted afterwards:

$$l\left\{u^2+\left(\frac{du}{dq_2}\right)^2\right\}-2u+\frac{1}{a}=0, \tag{24a}$$

$$lu=1+e\cos(q_2+\epsilon), \tag{24b}$$

where

$$l = a(1 - e^2), \qquad \left(\frac{1}{m} + \frac{1}{m'}\right)^2 k^2 = C(m + m')l. \tag{24c}$$

§ 13

Remarking upon the equation

$$a^3 n^2 = C(m + m'),$$

notice first that it is indispensable.

Writing it in the shape

$$an^2 = C(m + m')/a^2,$$

and applying it to circular orbits, it gave the first clue, in the hands of Robert Hooke, to the law of inverse square.

Notice too that k, which measures the angular momentum, and e, the eccentricity of the orbit, eliminate themselves.

The condition for oscillation, namely, that α, β should be real, requires that

$$(m\mu + m'\mu')^2 - 4\frac{m + m'}{mm'}k^2\gamma$$

should be positive ; this reduces to

$$a > a(1 - e^2);$$

consequently any eccentricity is permitted, from the circle ($e = 0$) to the parabola or straight line ($e = 1$).

But the equation appears to have nothing to do with any form ascribed to the orbit. It is purely a relation between mass, time and distance.

The obvious statement of the relation is that under a certain gravitational attraction, the two particles pulse in and out, at a certain mean distance, in a certain period.

All these quantities are separately unobservable ; yet there is a relation between them.

It is a relation between the scales adopted ; I see no prospect of abolishing it or simplifying it by counting from any different zero. This applies particularly to n, which measures the time.

"Centrifugal force," which apparently disappears because of the elimination of k, can be made to reappear if the equation is treated as by Robert Hooke.

It raises the question whether centrifugal force can be eliminated ; that is to say, whether stresses in one line are affected by cross-motions—for example, by rotation. It is a claim of General Relativity that they can be. In Euclidean space-time I see no prospect of doing so.

The matter is important, because we can set things in rotation, and the plane of rotation determines a direction which could not be changed arbitrarily. Newton—as subsequently many others—considered the matter, and concluded that there was a possibility—*causa non est prorsus desperata*. He performed an experiment with a bucket which was suspended

from a twisted cord and contained water; from this he concluded that relative rotation might be unobservable, while absolute rotation was observable.

The conclusion of Newton, that there was a possibility of assigning an absolute in direction, seems so directly contrary to impression that it will be referred to again in § 21.

The recognition of centrifugal force appears to me as essential as the recognition of singularities in mathematical expressions.

§ 14. *Solution for the General Case*

I approach the general case by methods analogous to those used in § 12.

t, t' now become distinct.

Note that Kepler's Laws of the fixed ellipse, etc., no longer hold when we incorporate terms depending on c^{-2}.

Since the co-ordinates have been limited by adoptions, it will be appropriate, mathematically, to work, not from O or O', but to define a certain other centre.

This is merely a mathematical device. The point so chosen need not have any other concrete importance.

In whatever way O, O' may determine a point, a second equation will be required to define it. This was taken in § 12 as marking the centroid of the two particles.

Choose in general this equation:

$$\frac{p_2}{q_1} = \frac{p_2'}{q_1'} = \frac{k}{r} = ku. \tag{25a}$$

Formally this is the same choice as (20*a*); if we write

$$P_2 = q_1^{-2}[P_2], \qquad P_2' = q_1'^{-2}[P_2'],$$

so that $[P_2]$, $[P_2']$ only differ from unity by small terms, and remember that $dq_2 = dq_2'$, it implies from § 4

$$\frac{q_1}{E'[P_2]} = \frac{q_1'}{E[P_2']} = \frac{1/u}{E'[P_2] + E[P_2']}, \tag{25b}$$

reducing to (20*b*) if c^{-2} is neglected.

Further take, in continuation of (17),

$$\Pi \equiv 1 - \frac{\mu_1 u}{c^2} + \frac{\mu_2 u^2}{c^4} \ldots,$$

$$P_1 \equiv 1 - \frac{\nu u}{c^2} + \ldots,$$

$$P_2 \equiv q_1^{-2}\left\{1 - \frac{\rho u}{c^2} + \ldots\right\} \equiv q_1^{-2}[P_2],$$

$$\Pi' \equiv etc., \qquad P_1' \equiv etc., \qquad P_2' \equiv etc. \tag{26}$$

§ 15

Before developing further, return to the equation of § 10,

$$dp_1 - dp_1' = -\left(\frac{\partial H}{\partial q_1} - \frac{\partial H}{\partial q_1'}\right)d\tau.$$

In $\frac{\partial H}{\partial q_1}$ we shall have a part dependent upon u as it appears explicitly, and this will equal the corresponding part in $\frac{\partial H}{\partial q_1'}$; these parts will therefore cancel on the right.

We shall also have a part in $\frac{\partial H}{\partial q_1}$ which depends on the differentiation of P_2 as far as the latter involves q_1^{-2} as a factor, which is in fact the "centrifugal force." This part is

$$\begin{aligned}\frac{\partial H}{\partial q_1} &= \ldots + c\frac{[P_2]}{Eq_1}\left(\frac{p_2}{q_1}\right)^2 \\ &= \ldots + c\frac{[P_2']}{E'q_1'}\left(\frac{p_2'}{q_1'}\right)^2, \qquad \text{by (25}a\text{), (25}b\text{)} \\ &= \frac{\partial H}{\partial q_1'}.\end{aligned}$$

Hence the right hand vanishes; and

$$p_1 - p_1' = const., \tag{27}$$

which is the equation of the "conservation of linear momentum."

Note that the integrals (27), (16), along with the adoptions (15a), (15b), fit the terminal equation (4b).

The proof above depends for its correctness upon the assumed developments

$$P_2 = q_1^{-2}\left\{1 - \frac{\rho u}{c^2} - \ldots\right\}, \qquad P_2' = etc.$$

The terms in { } are assumed to be produced by the gravitational field alone, since in the absence of one the factor q_1^{-2} is adequate. It seems impossible that a gravitational effect should depend on anything except u, which is the reciprocal of the distance that O ascribes to O', or O' to O.

Note that the terms in question, viz. "centrifugal force," must arise in the postulated form for H, though of course their values may be varied by other adoptions than those of (25a), (25b).

§ 16

Write the equations (25b)

$$q_1 = \phi/u, \qquad q_1' = \phi'/u,$$

and note that ϕ, ϕ' differ from the constants $m'/(m+m')$, $m/(m+m')$ only by terms factored by c^{-2}.

Use the equation (25a), together with those of § 4, to eliminate $d\tau$ in favour of dq_2, which is the same as dq_2'.

Then $$p_1 = \frac{[P_2]}{P_1} ku \cdot \frac{1}{q_1}\frac{dq_1}{dq_2} = \frac{[P_2]}{P_1} k\left(-\frac{du}{dq_2} + \frac{u}{\phi}\frac{d\phi}{dq_2}\right).$$

Substitute for p_1, p_2 in E; the expression becomes

$$E = c\left\{\Pi(mc)^2 + [P_2]k^2u^2 + \frac{[P_2]^2}{P_1}k^2\left(\frac{du}{dq_2} - \frac{u}{\phi}\frac{d\phi}{dq_2}\right)^2\right\}^{\frac{1}{2}}.$$

In developing this in descending powers of c, so as to include all terms as far as c^{-2}, note that Π must be taken as far as c^{-4}, but this is not necessary for P_1, $[P_2]$, ...

Developing, we have

$$E = mc^2 - \tfrac{1}{2}m\mu_1 u + \tfrac{1}{2}m\mu_2\frac{u^2}{c^2}$$
$$+\tfrac{1}{2}\left(1 - \frac{\rho u}{c^2}\right)\frac{1}{m}k^2u^2 + \tfrac{1}{2}\left(1 - \frac{2\rho - \nu}{c^2}u\right)\frac{1}{m}k^2\left(\frac{du}{dq_2} - \frac{u}{\phi}\frac{d\phi}{dq_2}\right)^2$$
$$-\frac{1}{8}\frac{m}{c^2}\left\{\mu_1 u - \frac{k^2}{m^2}(u^2 + u'^2)\right\}^2, \qquad (28)$$

where for brevity u' is written for $\dfrac{du}{dq_2}$.

These expressions may be substituted in the equation

$$const. = H = E + E'.$$

Define the constants to agree with the first order solution; *i.e.* taking as in (18)

$$H - (m+m')c^2 = -\tfrac{1}{2}(m\mu_1 + m'\mu_1')u + \tfrac{1}{2}k^2\left(\frac{1}{m} + \frac{1}{m'}\right)(u^2 + u'^2),$$

identify this with the equation of the ellipse

$$0 = \frac{1}{a} - 2u + l(u^2 + u'^2),$$

which requires

$$\frac{H - (m+m')c^2}{-1/a} = \frac{\tfrac{1}{2}(m\mu_1 + m'\mu_1')}{2} = \frac{\tfrac{1}{2}k^2\left(\frac{1}{m} + \frac{1}{m'}\right)}{l},$$

or

$$H = (m+m')c^2 - \tfrac{1}{4}\frac{m\mu_1 + m'\mu_1'}{a},$$
$$k^2 = \tfrac{1}{2}\frac{mm'}{m+m'}(m\mu_1 + m'\mu_1')l; \qquad (29)$$

to the same order, the value for E is

$$E = mc^2 - \tfrac{1}{2}m\mu_1 u + \frac{1}{2m}k^2(u^2 + u'^2)$$
$$= mc^2 - \frac{1}{4}\frac{m\mu_1 + m'\mu_1'}{m+m'}\frac{m'}{a} - \tfrac{1}{2}\frac{m^2\mu_1 - m'^2\mu_1'}{m+m'}u.$$
$$E' = m'c^2 - \frac{1}{4}\frac{m\mu_1 + m'\mu_1'}{m+m'}\frac{m}{a} + \tfrac{1}{2}\frac{m^2\mu_1 - m'^2\mu_1'}{m+m'}u. \qquad (30)$$

E, E' are not separately constant, unless $\mu_1/m'^2=\mu_1'/m^2$; but this is the final choice made in § 18.

§ 17

Having adopted these constants as in (29), consider the terms additional to $\frac{du}{dq_2}$ in the expression for H or $E+E'$; neglecting terms factored by c^{-4}, we have to consider the contribution made by ϕ, ϕ' to

$$\frac{1}{2m}k^2\left(\frac{du}{dq_2}-\frac{u}{\phi}\frac{d\phi}{dq_2}\right)^2+\frac{1}{2m'}k^2\left(\frac{du}{dq_2}-\frac{u}{\phi'}\frac{d\phi'}{dq_2}\right)^2.$$

The quantities ϕ, ϕ' are given by the equations (25b). Hence, from the actual expressions above,

$$\begin{aligned}\frac{1}{\phi}\frac{d\phi}{dq_2}&=\frac{1}{E'}\frac{dE'}{dq_2}-\frac{\rho}{c^2}\frac{du}{dq_2}+\frac{\rho m'+\rho' m}{m+m'}\frac{1}{c^2}\frac{du}{dq_2}\\&=\frac{1}{E'}\frac{dE'}{dq_2}+\frac{(\rho'-\rho)m}{m+m'}\frac{1}{c^2}\frac{du}{dq_2}.\end{aligned}$$

Hence the total contribution from

$$-\frac{1}{2m}k^2\cdot 2\frac{du}{dq_2}\cdot\frac{u}{\phi}\frac{d\phi}{dq_2}-\frac{1}{2m'}k^2\cdot 2\frac{du}{dq_2}\cdot\frac{u}{\phi'}\frac{d\phi'}{dq_2}$$

is zero.

In the last line of E, we may write for the term, in accordance with (30),

$$\begin{aligned}&-\frac{1}{8}\frac{m}{c^2}\left\{\mu_1 u-\frac{k^2}{m^2}(u^2+u'^2)\right\}^2\\&=-\frac{1}{8mc^2}\left\{\frac{m\mu_1+m'\mu_1'}{m+m'}\frac{m}{2a}-\frac{m^2\mu_1-m'^2\mu_1'}{m+m'}u\right\}^2.\end{aligned}$$

Thus the equation $H=E+E'=const.$, complete to the order c^{-2}, runs as follows, when we eliminate k^2 by (29), and divide throughout by

$$\tfrac{1}{4}(m\mu_1+m'\mu_1'),$$

$$\begin{aligned}0=&\frac{1}{a}-2u+l(u^2+u'^2)\\&+\frac{m\mu_2+m'\mu_2'}{m\mu_1+m'\mu_1'}\frac{u^2}{c^2}-\frac{m'\rho+m\rho'}{m+m'}l\frac{u^3}{c^2}-\frac{m'(2\rho-\nu)+m(2\rho'-\nu')}{m+m'}l\frac{uu'^2}{c^2}\\&-\frac{1}{2c^2}\frac{1}{m\mu_1+m'\mu_1'}\left[\frac{1}{m}\left\{\frac{m\mu_1+m'\mu_1'}{m+m'}\frac{m}{2a}-\frac{m^2\mu_1-m'^2\mu_1'}{m+m'}u\right\}^2+\frac{1}{m'}\left\{etc.\right\}^2\right].\end{aligned}\qquad(31)$$

It will conduce to simplicity if we substitute at once the values of μ_1, *etc.* finally chosen, viz.

$$\mu_1=\rho=\nu=2Cm'^2/(m+m'),\qquad \mu_1'=\rho'=\nu'=2Cm^2/(m+m')$$

$$\mu_2=-\mu_1\kappa c^2,\qquad \mu_2'=-\mu_1'\kappa c^2,\quad \text{if}\quad \kappa c^2=\frac{m\mu_1'+m'\mu_1}{m+m'}=2C\frac{m^3+m'^3}{(m+m')^2}.\qquad(32)$$

Some remarks upon the choice will be made later.

It will be noticed that the substitution employed in General Relativity, viz.

$$u = w + \lambda w^2/c^2,$$

which is equivalent to

$$r = r_1 - \lambda/c^2,$$

is not required. I should in any case regard it as a dubious resource, since the effect of making an addition $-\lambda/c^2$ to a distance is not the same at both ends of a radius.

Write the consequent equation

$$0 = \frac{1}{a+\Delta a} - 2u + l(u^2+u'^2) - \kappa u[u + l(u^2+u'^2)]$$

where

$$\Delta a = +\frac{1}{4}\frac{C}{c^2}\frac{mm'}{m+m'}.$$

Multiply throughout by $1+\kappa u$, neglecting c^{-4} :

$$(l-3\kappa)u^2 + lu'^2 - \left(2 - \frac{\kappa}{a}\right)u + \frac{1}{a+\Delta a} = 0\,; \tag{33}$$

differentiate with respect to u :

$$l\frac{d^2u}{dq_2^2} + (l-3\kappa)u - \left(1 - \frac{\kappa}{2a}\right) = 0. \tag{33a}$$

This is the equation of a "rotating ellipse." The solution is

$$(l-3\kappa)u = 1 - \frac{\kappa}{2a} + \epsilon\cos(pq_2+\delta)$$

where ϵ, δ are arbitrary constants, provided

$$p^2 = 1 - 3\kappa/l,$$

that is to say, q_2 must execute more than 2π in order to secure a repetition of u, in the ratio $3\kappa/2l$, or

$$\frac{3C}{lc^2}\frac{m^2 - mm' + m'^2}{m+m'}. \tag{34}$$

In order to compare with General Relativity, put $m' = 0$; the amount is

$$\frac{3Cm}{c^2 l}.$$

This is precisely the same quantity as is calculated by General Relativity.

Before discussing the choice of μ_1, . . . in (32), consider the form taken by Kepler's Third Law in this case. Eliminate q_2 from (33) by means of the equation (25), *i.e.* reverse the process of § 16.

We get an equation

$$\frac{1}{k^2}\left(\frac{mm'}{m+m'}\right)^2 l\left(\frac{dr}{d\tau}\right)^2\left(1-\frac{1}{2}\frac{\kappa}{a}+\frac{2\kappa}{r}\right)=-(l-3\kappa)\frac{1}{r^2}+\left(2-\frac{\kappa}{a}\right)\frac{1}{r}-\frac{1}{a+\Delta a}.$$

Substitute for k^2 from (29); divide throughout by $1-\kappa/2a$, and multiply by r^2:

$$\frac{1}{C(m+m')}\left(\frac{dr}{d\tau}\right)^2 r(r+2\kappa)=-\left(l-3\kappa+\frac{\kappa l}{2a}\right)+2r-\frac{1+\kappa/a}{a+\Delta a}r^2.$$

We may write this

$$\frac{1}{C(m+m')}\left(\frac{dr}{d\tau}\right)^2(r+\kappa)^2=-(l+\delta l)+2r-\frac{1}{a+\delta a}r^2\equiv\gamma(\alpha-r)(r-\beta)$$

provided

$$\begin{aligned}\gamma\alpha\beta&=l+\delta l,\\ \gamma(\alpha+\beta)&=2,\\ \gamma&=1/(a+\delta a).\end{aligned}$$

This is the same equation as before; r oscillates from β to α, and back again; write

$$r+\kappa=(\alpha+\kappa)\sin^2\phi+(\beta+\kappa)\cos^2\phi$$

or

$$r=\alpha\sin^2\phi+\beta\cos^2\phi;$$

then

$$r\Big|_{\beta}^{\alpha}=\phi\Big|_{0}^{\frac{\pi}{2}}.$$

Call T the "period"; put $T=2\pi/n$; the equation leads to

$$n^2(a+\Delta a-\tfrac{1}{6}\kappa)^3=C(m+m'),\tag{35}$$

where Δa, κ are the quantities already so denoted. The eccentricity e, and the moment of momentum, k, eliminate themselves as before.

Note that τ is involved, which is not an observable.

§ 18

I resume the question of the choice of μ_1, . . . given in (32).

If all these quantities are carried through the equation as unknowns, I find that λ makes no difference in the motion of the apse, and

$$1-p^2=-\frac{1}{2lc^2}\frac{m\mu_2+m'\mu_2'}{m\mu_1+m'\mu_1'}+\frac{1}{2lc^2}\frac{m'(\rho+\nu)+m(\rho'+\nu')}{m+m'}+\frac{1}{4lc^2}\frac{(m^2\mu_1-m'^2\mu_1')^2}{(m+m')mm'(m\mu_1+m'\mu_1')}.\tag{36}$$

The one datum we possess for evaluating this expression is the rate of progress of the perihelion of Mercury, showing a fault in Kepler's Law of the fixed ellipse. Accepting the data as substantially correct, let us aim to

make this the same value as that calculated by General Relativity, or say, following § 16, let us say the expression above

$$= +\frac{3C}{lc^2}\frac{m^2 - mm' + m'^2}{m + m'}. \tag{34}$$

The equation is indeterminate, mathematically. The values recorded in (32) represent only the first simple ones at which I arrived.

The vital matter is the choice for μ_1, μ_1'.

By (19) it is seen that if c^{-2} is neglected, the coefficient in the energy is

$$H = \ldots - \tfrac{1}{2}(m\mu_1 + m'\mu_1')u \ldots$$

This coefficient must be identified with what experience assigns.

The question concerned is the celebrated identity of inertial and gravitational mass.

To simplify ideas, let m' be inconsiderable compared to m. Then the first term vanishes. We may think of m' as an ordinary body at the Earth's surface. The second term contains m' as a factor, multiplied by μ_1' which measures the gravitational effect of the Earth. Hence no ordinary test, as by moving bodies about, or comparing the forces of springs and weights, or in any other way, will reveal the gravitational effect of a mass.

It is necessary to perform the Cavendish experiment or some similar experiment, *i.e.* to ascertain directly the gravitational effect of two masses, and to determine the same in other units.

The Cavendish experiment is a very difficult experiment, so that its adequate performance by C. V. Boys in 1895 remains a feat. I would only make this remark, that behind his work is the assumption that Newton's formula for the attraction of two spheres is exact. Nor presumably could the experiment have been done otherwise, since balls that are known to be homogeneous spheres are difficult things to make, and particles are impossible.

The Cavendish experiment assumes Newton's theory to be correct, and Newton's theory is based upon Kepler's Laws. He generalized his theory afterwards "by induction," making it a universal property of matter.

None of the tests that can be put to it, as the lunar theory, precession and other astronomical phenomena, or the Cavendish experiment, though impressive collectively, amount to much in stringent definition. It is more convincing that Newton, with his unequalled sense of logic, judged it to be tenable.

But Einstein, with perhaps a comparable sense of fitness, has given a different description, which covers the one point where Kepler's Laws are violated. See some remarks in § 21 upon this point.

Without attempting to decide the question, one purpose of the present paper is to show that certain peculiarities introduced by Einstein are superfluous.

Take then Newton's view, that the gravitational effect of any particle is directly proportional to the parameter m, introduced in § 6. "If A varies as B when C is constant, and varies as C when B is constant, then A varies

as BC when both B and C vary." Accordingly we may write, as is usually done, the gravitational energy of two particles O, O' as

$$const. - Cmm'u,$$

where C is some constant changing with the units. Equate this to the theoretical expression found above :

$$m\mu_1 + m'\mu_1' = 2Cmm'. \tag{37}$$

This is the only equation that μ_1, μ_1' have to satisfy. Strictly speaking, it is indeterminate. Regarding μ_1, μ_1' as Cartesian co-ordinates, the equation restricts them to a straight line ; any point between any two solutions will serve. I note the three cases as being fairly simple :

$$\begin{array}{lll} \text{(A)} & \mu_1 = Cm', & \mu_1' = Cm, \\ \text{(B)} & \mu_1 = 2Cm'^2/(m+m'), & \mu_1' = 2Cm^2/(m+m'), \\ \text{(C)} & \mu_1 = 2Cmm' = \mu_1'. & \end{array} \tag{38}$$

Here

$$2(\text{A}) = (\text{B}) + (\text{C}).$$

The solution (A) represents the ordinary theory.

I choose the set (B), my reason being that I have found a consequent choice of the other quantities λ, ν, . . . which is satisfactorily simple, whereas I have not found a simple set for (A) or (C).

I add a remark about simplicity. All ultimate relations, as the equation $E = mc^2$, are simple. But I do not put forward these results as ultimate. The simplicity I seek is such as I can handle, my mind soon losing itself in complications.

The solution (32) was found by trial. Any better solution than (32) would have preferable claims.

Confine attention to the solution (B). If $m'/m \to 0$, $\mu_1' = 2Cm$, compared with Cm of ordinary theory. In the same way twice the mass figures in General Relativity. Or alternatively $2C$ figures where (A) would give C only. Since both are solutions of equation (37), which expresses the result of the Cavendish experiment, the change produces no effect at all, in the calculated mean density of the Earth or the quantities derived from it such as the mass of the Sun in grams, the masses of the planets, and so forth. The sole effect is in the value ascribed to the Constant of Gravitation.

§ 19. *Gravitational Deflection of Light*

Since this paper is confined to the theory of two particles of matter, the transmission of light may be considered outside its scope. But light and other radiations occupy such a fundamental position that this would be unsatisfactory. I give then the corresponding treatment as is given in General Relativity, showing that the same results may be calculated from the procedure of the present paper.

I would say, however, that I do not regard it seriously as throwing any light upon the mechanical relations of a photon.

Assume that a photon moves with velocity c when the gravitational field is insensible. By "velocity" must be understood, not $\left\{\left(\frac{dq_1}{d\tau}\right)^2 + \ldots\right\}^{\frac{1}{2}}$, but the local velocity $\left\{\left(\frac{dq_1}{dt}\right)^2 + \ldots\right\}^{\frac{1}{2}}$, as in General Relativity. This is in accordance with the view that c, the velocity ascertained by experiment, is expressed by $dq_1, dt, \ldots$

Let $O(m)$ be the photon, and $O'(m')$ the Sun.

By (30), E is constant; by (9), $p_\infty = mc$; hence

$$E = c\{m^2c^2 + p_\infty{}^2\}^{\frac{1}{2}} = mc^2\sqrt{2}.$$

But in the neighbourhood of the Sun, writing $P_1 = (P_2) = \Pi$ as in (32),

$$(E/c)^2 = \Pi(mc)^2 + \Pi(mv)^2.$$

Substitute for E the value just found, and for Π the value $1 - 2Cm'u/c^2$; we get

$$v^2 = 2(1 + 2Cm'u/c^2) - 1 = 1 + 4Cm'u/c^2.$$

If we write, as usual,

$$v^2 = k^2(u^2 + u'^2),$$

this equation runs

$$\frac{c^2k^2}{2Cm'}(u^2 + u'^2) = 2u + \frac{c^2}{2Cm'}, \tag{39}$$

which represents an hyperbola, described about the inner focus, of semi-major axis $2Cm'/c^2$, as in the theory of General Relativity. But notice that whereas that theory makes the velocity of the photon decrease as it moves from infinity, the formula above makes it increase. The result is the same as relates to the deviation of the particle from rectilinear motion.

If the investigation is followed out on the Newtonian theory, it will be found that the same total energy assigns a semi-major axis of just half the amount, as it is well known to do.

We may mention also the case of the shift of the spectral lines towards the red in the Sun, and still more, in stars of greater density. This is shown by noticing that the gravitational field introduces a factor Π in $\Pi(mc)^2$, which may be stated otherwise as a reduction in the velocity of light in the ratio $\Pi^{\frac{1}{2}} : 1$, or $1 - Cm'u/c^2$. Again this is the same result as was given by Einstein.

§ 20

With the choices made in (32), we may write

$$E = mc^2\Pi^{\frac{1}{2}}\{1 + v^2/c^2\}^{\frac{1}{2}}, \tag{40}$$

where

$$\Pi = 1 - \frac{\mu_1 u}{c^2} + \frac{\mu_2 u^2}{c^4} \ldots$$

and

$$\mu_1 = \frac{2Cm'^2}{m+m'},$$

$$\mu_2 = -\mu_1 \cdot \kappa c^2 = -\mu_1 \cdot 2C\frac{m^2 - mm' + m'^2}{m+m'},$$

or

$$\Pi = 1 - \frac{\mu_1/c^2}{r-\kappa}.$$

The value of κ for the Earth, as m', is about $\frac{1}{2}$ cm.; that for the Sun is about 1·5 km.

We may confront this expression with that found in § 6, viz.

$$E = mc^2\Pi^{\frac{1}{2}} \Big/ \left[1 - \frac{1}{c^2 P}\left(\frac{dq}{d\tau}\right)^2\right]^{\frac{1}{2}}.$$

The expressions belonging to Π, P are not material, since they relate to the body $O'(m')$.

The question arises whether $\left\{1 + \frac{1}{c^2}\left(\frac{dq}{dt}\right)^2\right\}^{\frac{1}{2}}$ and $\left\{1 - \frac{1}{c^2}\left(\frac{dq}{d\tau}\right)^2\right\}^{-\frac{1}{2}}$ are consistent.

Both are derived from the same definition of E; their consistency may be verified through the equation (5),

$$c^2 d\tau^2 = c^2 dt^2 + dq^2.$$

The question further arises whether they agree with experience. For any actual value found in a particle of matter, the ratio of its velocity to that of the velocity of light is a small fraction; the two expressions give sensibly the same quantity, and agree with experience.

Large velocities, exceeding half the velocity of light, have not been realized, except in the emission of particles which are certainly not particles of matter.

They do not enter in this discussion.

If it is concluded, for example, that the Bucherer experiment verifies a form containing $(1 - v^2/c^2)^{-\frac{1}{2}}$ to the exclusion of any other, where v is the local velocity, then it would follow that form (1), which we have been investigating, was not appropriate to the problem. But the Bucherer experiment concerns the emission of electrons, and so is outside the scope of the present paper.

We remark that the treatment of the photon in § 19, resulting in the equation $p = mv = mc$, and consequently $E = mc^2\sqrt{2}$, implies $dq/d\tau = c/\sqrt{2}$ only, and does not admit $dq/d\tau$ to range up to equality with c, which would give $p = \infty$. It is all highly artificial.

§ 21. *Bearing upon Philosophy*

By the title of this paper, and the scattered remarks in the preceding pages, it will be gathered that in so general a matter it is impossible not to have the epistemological bearing in mind.

The paradox that we do not know if an external world even exists is generally held to be an overstatement. In order to argue about it, we must assume that we have *some* knowledge outside ourselves. It does not follow that this knowledge is clear or complete.

A postulate of science is that it is possible to render this knowledge clear. At least in parts it can be rendered logical. How far the parts must extend depends upon the scope attempted.

Logic, as I understand it, is no more than a device for producing "knowledge" where there is none, say elsewhere, or in the future. Its fundamental assertion is that if B has been found to follow A at a sufficient number of points in space-time, then B will follow A at another point in space-time—about which we know nothing.

Mathematics is merely a systematized logic. The system reduces its scope. Mathematics is exact and complete, whereas most of our knowledge is fragmentary and hazy, or even in parts it is contradictory. We must then ask, Over what regions does mathematics correspond with knowledge?

To confine the question to the subject of this paper, space, time and matter suggest themselves for inquiry. Can we construct a logical system that will correspond to our knowledge or impressions of these?

It is assumed in the previous pages that our source of knowledge of the external world is by the senses—the "gateways to knowledge" as Lord Kelvin called them. I believe that anything that goes beyond the senses may be useful for trial, but must be held tentatively only. For example, the theory of the atom, or any statement regarding the future.

I shall not examine how many senses there are, nor whether they separate from one another, nor how they perform.

The relevant question is, Whether the concepts described below would really fall under sense perception?

Perception of the momenta, p, is supposed to be derived from the muscular sense.

The elements of the co-ordinates, dq, is supposed to be derived from a comparison of the present position of the incident in space-time, with that supplied by memory of the immediately preceding position.

The way in which the integral values of the co-ordinates q is built up has been described in § 7.

For each incident we thus have a four-dimensional manifold twice given by the senses, and therefore we may expect some relations between the two to appear in experience.

If this is so, it would follow that many of the laws of nature, for example, all laws of motion, were merely products of our own way of viewing the facts. This would answer in part Newton's question, "Whence arises all that Order and Beauty that we see in the World?"

The answer is: It arises from rhythm, due to the observation of the same fact by different avenues.

To return to the question, whether mathematics supplies a tenable description of our experience, we have the fact that any "geometry"

assumes—what is not the case—that we have some absolute state of affairs presented to us, by which to measure our experience.

This conception is suggested by the practically instantaneous character of sight of ordinary objects.

To clear ideas, we may imagine any experiment to be performed with eyes bandaged and with one finger as explorer. It will then be clear that each datum will reach us mixed with an irrelevant part of what has happened during the time taken to move the finger from one point to another. If we attempt a repetition we may expect some of these data to change.

Under such circumstances, if the observer makes an account of what he perceives, it is most unlikely that he would frame it upon a geometrical background.

It might appear that the variety of senses, and the practically instantaneous character of sight for close objects modified this. But variety of manners of approach does not increase the number of objects contemplated, and we know of some phenomena, like the stars and nebulæ, where mere transmission absorbs enormous periods, far exceeding any span of our own.

It is clear too that this is not an overestimate of the obscurities, since any knowledge we possess must pass through the sieve of our own prejudices, nor do we even know that another person looks upon the same facts in the same way.

We have no reason to call *any* datum absolute. There appears no reason to assign any absolute quality to τ.

To talk even of "change" postulates some absolute knowledge which does not change.

There is a complete breakdown of all statements, if time, as we perceive it, is not treated exceptionally, but may be amalgamated with the co-ordinates. To verify this, we need only try to convert any statement in this section into a timeless form.

Even the simplest mathematical step may be far from innocuous. If we take it that O has some knowledge of O', in a Euclidean manifold this implies ascription of a direction to the line OO', and this again, as we have seen in § 13, implies knowledge of an absolute direction.

The familiar conclusion of § 10, that two bodies continue to move in one plane, equally demands it. But what *is* a plane? A plane implies a certain choice of co-ordinates. And by what tests do we know that it is fixed?

As we cannot admit the knowledge of an absolute direction, we are forced to exclude even the simplest mathematical statement regarding another body.

We are reminded that it is illegitimate to claim that O has any precise knowledge of O'.

In non-Euclidean geometries, it does not appear to me that a curvature of space-time will alter this in principle ; for though a "straight line," now a geodesic, is twisted so as to become a Keplerian ellipse, it appears to me to possess a definite direction of its own at any point.

I admit that a great gain in generality is achieved by abolishing all

reference to planes and axes ; but in common with a number of other people, I fear my senses are too deeply rooted in such non-final descriptions, and I am not willing to transcend them.

It is a claim of General Relativity that it explains the identity of inertial and gravitational mass otherwise than by the mere statements that "inertia is a property of matter," and "gravitation is a property of matter."

We have the fact of experience that a body A produces in a body B at a different point, a change of motion in B, which is proportional to a certain parameter of A. General Relativity secures this by attributing to A a certain "curvature of space-time," without saying how this curvature imposes a change of motion upon B. The older statement merely postulated "action at a distance," which was a confession that we know nothing of the circumstances. It appears to me that this is more consonant with modern practice, which, when it knows nothing, endeavours to say nothing. A curvature of space appears to me an attempt to rely upon a "model," for we *can* follow, in a vague way, a curvature of space. Personally I regard models as useful, even though they are confessedly imperfect. Indeed I regard the imperfection as an advantage, because it disclaims finality.

Summary

Postulating Euclidean space-time, and the energy of a particle

$$E = c\sqrt{(\Pi\pi^2 + P_1 p_1^2 + P_2 p_2^2 + P_3 p_3^2)},$$

where π, p_1, p_2, p_3 are momenta, conjugate to displacements cdt, dq_1, dq_2, dq_3 and the coefficients Π, P_1, P_2, P_3 are functions of the co-ordinates and include the effect of a gravitational field, the case is discussed of two particles, related by the equation $E + E' = H$. It is shown that an equation

$$p_1 dq_1 + \ldots + \pi c dt + p_1' dq_1' + \ldots + \pi' c dt' - H d\tau = 0$$

obtains identically, also $H = const.$, and that if t does not occur explicitly we may put $\pi = mc$. Further, the complete solution for two particles in space-time is given, and the results shown to be the same as those of General Relativity, if we put

$$E = mc^2\left(1 - \frac{2C}{c^2}\,\frac{m'^2 u_1}{m + m'}\right)^{\frac{1}{2}}(1 + v^2/c^2)^{\frac{1}{2}},$$

where m, m' are the masses of the two particles, u_1 differs slightly from the reciprocal of the distance between the particles, and v is the local velocity. It is concluded that the curvature of space-time is not an essential.

FLUORESCENCE IN H_α AND H_β.

R. v. d. R. Woolley, Ph.D.

§ 1. Until recently it has been impossible to account for three features of astrophysical spectra which seem to bear a relation to one another, namely, the presence of bright lines in certain stellar spectra, the presence of bright reversals in certain lines in some spectra, and the high central intensities of lines in general. The ordinary * equations of transfer of radiation in the outside of a star lead to a dark or absorption line, which should have a very low central intensity. The observed central intensities are of the order 10^{-1} of the intensity of the continuous background, and the theory gives central intensities of order 10^{-2} at the most generous estimate of the effect of collisions.† It is clear that the mechanism which is the cause of bright-line emission is likely to increase the central intensities of absorption lines, but it was only when Rosseland ‡ pointed out that cyclic quantum transitions are not equivalent to macroscopic selective absorption of radiation that any clue to the explanation of the formation of bright lines was unearthed. Rosseland proved an elegant theorem which states that if three quantum states are cyclically connected by possible electron transitions (of which, let us say, one is in the ultra-violet and two are in the red end of the spectrum), and if the atom is exposed to dilute temperature radiation (by which is meant radiation whose frequency distribution is that of black-body radiation at some temperature T_1 and whose radiation density is that of black-body radiation of some temperature T_2 less than T_1), then the cyclic changes will proceed more often in one sense than in the other, namely, in the sense that absorbs one quantum in the ultra-violet and emits two in the red. Atoms exposed to dilute temperature radiation will fluoresce, which is to say, that they will absorb more ultra-violet quanta than they emit and emit more visible quanta than they absorb. This is not to be regarded as a singular property of the atom, that it fluoresces in the presence of dilute temperature radiation, but as a property of dilute temperature radiation, that it makes atoms fluoresce.

Unfortunately no triangular cyclic transition can exist without incorporating at least one forbidden transition. Let the first state have the inner quantum number j. Then the second state must have the inner quantum number $j \pm 1$ and the third state either $j+2$, j or $j-2$, so that the third

* By "ordinary" equations is meant those equations developed by Schwarzschild, Milne and Eddington, which ignore transfer of radiation from one frequency to another and are equivalent to a single quantum transition between two states with no alternative transitions from either state. Milne (*M.N.*, **89**, 17, 1929) specifically mentions the Rosseland cycle as the probable cause of bright-line spectra, and Eddington (*ibid.*, p. 620) specifically mentions the necessity of considering alternative transitions.

† Thackeray, *Observatory*, **56**, 228, 1933.

‡ *Ap. J.*, **63**, 218, 1926.

transition which completes the cycle must violate the rule $\Delta j = \pm 1$. Any closed cycle must contain at least four transitions, and the fluorescence is weaker than for the triangular course. But Rosseland's principle can be extended to the process known as recombination. According to Rosseland's argument, when a neutral atom is exposed to dilute temperature radiation it is more likely to be ionised from the ground state and to recombine with an electron to form a neutral atom in a high state of excitation, from which state the atom will get to the ground state by emitting quanta in its line spectrum, than it is likely to be ionised from the excited state and recombine in the ground state. There will accordingly be a fluorescence and the triangular cycle is not forbidden by selection rules. This process of recombination is now generally recognised as forming the basis of the explanation of bright-line emission in B-type stellar spectra, but writers who speak of recombination taking place in such a way as to form neutral atoms in a state of high excitation, rather than in the ground state, do not always make the position clear. In thermodynamical equilibrium, just as many recombinations occur in the ground state there occur ionisations of atoms in this state, by the principle of detailed balancing. The same applies to the excited states. Why, then, should we expect an *excess* of recombinations in the higher states in a stellar atmosphere? Because the radiation is dilute temperature radiation, and, from Rosseland's theorem, the atmosphere will fluoresce.

§ 2. The mathematical treatment of this problem is extremely simple. Suppose that there are m connected states, including the ionised state, and let A_{ik} be the probability of a transition from state i to state k so that if n_i is the number of atoms per c.c. in state i then $n_i A_{ik}$ is the number of transitions from state i to state k per c.c. per second. For each state there will be an equation expressing the fact that the number of transitions which end in this state per second is equal to the number of transitions which start from it, or, for state 1,

$$n_2 A_{21} + n_3 A_{31} + \ldots + n_m A_{m1} = n_1 (A_{12} + A_{13} + \ldots + A_{1m}) \quad (2.11)$$

or, if we like to write $A_{11} = -(A_{12} + A_{13} + \ldots + A_{1m})$,

$$n_1 A_{11} + n_2 A_{21} \quad + \ldots \quad + n_m A_{m1} = 0. \quad (2.12)$$

There are m such equations and $m - 1$ unknowns (the ratios $n_2/n_1, \ldots, n_m/n_1$). The determinant

$$\begin{vmatrix} A_{11} & A_{21} & A_{m1} \\ A_{12} & A_{22} & A_{m2} \\ & & \\ A_{1m} & A_{2m} & A_{mm} \end{vmatrix} \quad (2.13)$$

must accordingly vanish and the values of $n_1, n_2, \ldots, n_m$ will be proportional to the minors $\mathfrak{A}_{11}, \mathfrak{A}_{21}, \ldots, \mathfrak{A}_{m1}$ of the first row (or to the minors of any row). To determine n_2/n_1, etc., all that has to be done is to find out the values of the transition coefficients A_{ik} and work out the minors in a determinant. Nothing could be simpler: but unfortunately there is an infinite number of possible states in the hydrogen atom and the transition coefficients have only been calculated for a few of them. The complete solution of the problem is

not feasible, but we can get approximate solutions by considering a limited number of states.

Using the notation of Eddington's *Internal Constitution of the Stars* we have for A_{21} and A_{12}

$$A_{21} = b_{21} + a_{21}\rho_\nu, \qquad A_{12} = a_{12}\rho_\nu,$$

ρ_ν being the energy density of the radiation. If $\rho_{\nu,\,T}$ is the equilibrium value of ρ_ν at a temperature T, then

$$a_{21}\rho_{\nu,\,T} = b_{21}/e^{h\nu/RT} - 1 \quad \text{and} \quad a_{12}\rho_{\nu,\,T} = \frac{q_2}{q_1} \times b_{21}/e^{h\nu/RT} - 1,$$

where q_2, q_1 are the statistical weights of states 2 and 1 respectively. The coefficients b_{21} are known for certain hydrogen transitions.* In the case of dilute temperature radiation corresponding, in colour distribution, to a temperature T, and having a density $\rho_\nu = W \times \rho_{\nu,\,T}$, such as will obtain at a point outside the surface of a star at which the disc subtends a solid angle $\Omega = 4\pi W$, we have

$$A_{21} = b_{21}\{1 + W/(e^{h\nu/RT} - 1)\} \quad A_{12} = q_2/q_1 \times W b_{21}/(e^{h\nu/RT} - 1) \qquad (2.2)$$

and the coefficients A_{ik} can be calculated from the tabular b's. An exception occurs in the case of the ionised state (which we refer to as state 5), for which no Einstein coefficients exist. The number of transitions from any state to state 5 can be found from the oscillator strengths f, which are known.†

In the case of a line transition

$$f_{12} = \tfrac{1}{3} \times q_2/q_1 \times \frac{b_{21}}{\omega},$$

where ω is the "classical damping constant" equal to $8\pi^2 e^2 \nu^2/3mc^3$. The coefficient of absorption s_ν is equal to

$$\frac{\pi e^2}{m} \times \frac{f\delta}{(\nu - \nu_0)^2 + \delta^2},$$

where δ is the half breadth of the line (equal to $\omega/4\pi$ on the classical theory).‡ The total amount of radiation absorbed per atom per second is $\int s_\nu \rho_\nu d\nu$ which is equal to $\pi e^2/m \times f\rho_\nu$ and the number of quanta absorbed per second, or A_{12}, is $\pi e^2/mh\nu \times f\rho_\nu$. After a little reduction we get

$$A_{12} = \frac{3 f_{12}\omega}{e^{h\nu/RT} - 1}, \qquad (2.3)$$

and it can easily be verified that since $f_{12} = 1/3 \times q_2/q_1 \times b_{21}/\omega$ we have

$$n_1 A_{12} = n_2 A_{21}.$$

Values of A_{15}, A_{25}, etc., can be calculated immediately from (2.3).

The reverse quantities A_{51}, A_{52}, etc., can be calculated from the principle of detailed balancing, since in thermodynamical equilibrium $n_1 A_{15} = n_5 A_{51}$. Of course, A_{51} is not an atomic constant like A_{21}, it is the number of electron captures per ion per sec. which result in the formation of an atom in state 1,

* F. G. Slack, *Phys. Rev.*, **31**, 527, 1928. Sugiura, *Journ. de Phys.*, **8**, 113, 1927; *Sci. Papers Inst. Tokyo*, **11**, 1, 1929. † *Handbuch der Physik*, **24** (1 Teil), 443.

‡ Weisskopf, *Observatory*, 56, 302, 1933.

and it depends on the electron pressure. But as long as we do not want to calculate n_5/n_1 there is no need to specify an electron pressure. All that we are concerned with is the way in which the captures are distributed among the states 1 to 4. We suppose that the proportions are those which obtain in thermodynamical equilibrium at the temperature considered. Since there is no secular increase in the ionisation, the number of ionisations per c.c. per sec. is equal to the number of recombinations, or

$$n_1A_{15}+n_2A_{25}+\ldots+n_5A_{55}=0$$

and we can treat state 5 just like any other state, the number of atoms in this state being arbitrary, and the values of A_{51}, etc., being found from

$$A_{51}:A_{52}:A_{53}:A_{54}::n_1A_{15}:n_2A_{25}:n_3A_{35}:n_4A_{45}.$$

If the ratios $A_{51}:A_{52}$, etc., were found to depend sharply on temperature, we should have to consider very carefully what we meant by "temperature," if

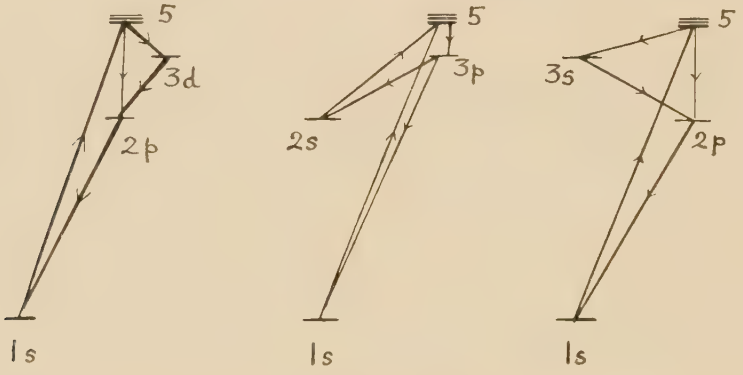


FIG. 1.—*Transition schemes for* H_α.

the temperature as judged by electron velocities did not agree with the local radiation temperature; but the ratios are practically independent of temperature over the range 6000° – 7000°. It is, of course, the electron-velocity-temperature that governs the captures.

§ 3. *The Central Intensities of* H_α *and* H_β *in the Solar Spectrum*

The effective temperature of the solar radiation, when allowance has been made for absorption lines, is about 6000°.* The colour temperature is considerably higher. H. H. Plaskett† gives about 6500° for the colour temperature of integrated sunlight. The radiation at the surface of the Sun is dilute, firstly because there is no inflowing radiation, and secondly on account of the excess of colour temperature over effective temperature. The value of $1/W$ is $2\times(65/60)^4$ or, approximately, 3. We investigate the fluorescence of H_α and H_β by considering the simplest possible transition schemes involving four levels only in each case. The diagram of the transition schemes for H_α is shown in fig. 1. The three cases are considered separately.

* *Observatory*, 57, 13, 1934. † *Pub. D.A.O.*, 2, 213, 1923.

We take $W = 1/3$ and $T = 6500°$. The constants used are shown in Tables I and II.

TABLE I

Absorption and Emission to Continuum for $T = 6500°$ and $W = 1/3$

State	ν, in cm.$^{-1}$	$e^{h\nu/RT} - 1$	ω	f	A_{n5}	A_{5n}
$1s$	109 678	$3{\cdot}08 \times 10^{10}$	$2{\cdot}67 \times 10^{9}$	0·436	0·0378	113
$2s$	27 420	418	$1{\cdot}67 \times 10^{8}$	0·362	$1{\cdot}45 \times 10^{5}$	17·9
$2p$				0·196	$7{\cdot}87 \times 10^{4}$	9·67
$3s$	12 187	13·7	$3{\cdot}29 \times 10^{7}$	0·293	$7{\cdot}06 \times 10^{5}$	5·04
$3p$				0·217	$5{\cdot}23 \times 10^{5}$	3·64
$3d$				0·100	$2{\cdot}41 \times 10^{5}$	1·72
$4s$	6 856	3·53	$1{\cdot}04 \times 10^{7}$	0·248	$7{\cdot}32 \times 10^{5}$	1·48
$4p$				0·214	$6{\cdot}32 \times 10^{5}$	1·39
$4d$				0·149	$4{\cdot}40 \times 10^{5}$	0·97

TABLE II

Line Transitions for $T = 6500°$ and $W = 1/3$

Line	ν in cm.$^{-1}$	$e^{h\nu/RT} - 1$	$b_{21} \times 10^{-7}$	f	A_{21}, etc.	A_{12}, etc.
$1s$–$2p$	82 258	$7{\cdot}35 \times 10^{7}$	63·2	0·416	$6{\cdot}32 \times 10^{8}$	8·59
$1s$–$3p$	97 491	$2{\cdot}11 \times 10^{9}$	16·9	0·079	$1{\cdot}69 \times 10^{8}$	0·080
$1s$–$4p$	102 823	$6{\cdot}81 \times 10^{9}$	6·8	0·029	$6{\cdot}8 \times 10^{7}$	0·010
$2s$–$3p$	15 233	27·6	2·26	0·425	$2{\cdot}28 \times 10^{7}$	$8{\cdot}19 \times 10^{5}$
$2p$–$3s$			0·632	0·014	$0{\cdot}648 \times 10^{7}$	$2{\cdot}64 \times 10^{4}$
$2p$–$3d$			6·47	0·694	$6{\cdot}55 \times 10^{7}$	$1{\cdot}30 \times 10^{6}$
$2s$–$4p$	20 565	91·6	0·975	0·102	$0{\cdot}978 \times 10^{7}$	$1{\cdot}06 \times 10^{5}$
$2p$–$4s$			0·258	0·003	$0{\cdot}259 \times 10^{7}$	$3{\cdot}14 \times 10^{3}$
$2p$–$4d$			2·07	0·122	$2{\cdot}08 \times 10^{7}$	$1{\cdot}26 \times 10^{5}$

The number of downward transitions per second is $n_3 A_{32}$ and the number of upward transitions is $n_2 A_{23}$. These exactly balance in thermodynamical equilibrium. A_{32} is an atomic constant, but A_{23} is proportional to the intensity of radiation of frequency ν_{23} and is proportional to W. The ratio of downward to upward transitions is $n_3 A_{32}/n_2 A_{23}$, which we will call α. Choosing the minors $\mathfrak{A}_{35}$, $\mathfrak{A}_{25}$ in the bottom row we have

$$\alpha = \frac{\mathfrak{A}_{35} A_{32}}{\mathfrak{A}_{25} A_{23}}.$$

If there were no alternative transitions giving rise to fluorescence—if the atom had only the two states 2 and 3—we should have the same balancing as actually obtains in thermodynamical equilibrium and α would be equal to unity.

In order to evaluate α it is necessary to calculate the minors $\mathfrak{A}_{35}$, $\mathfrak{A}_{25}$ in the determinant (2.13). A short cut to the evaluation is afforded by the fact that some terms are zero while others are negligible. Working out the strongest transition for H_α ($2p$–$3d$) we have $A_{13} = A_{31} = 0$, and also that $A_{32} >> A_{35}$, so that $A_{32} = -A_{33}$ very nearly; similarly $A_{21} = -A_{22}$. Then

$$\mathfrak{A}_{25} = A_{32}(A_{12} + A_{15})(A_{52} + A_{53}) + A_{12} A_{32} A_{51},$$
$$\mathfrak{A}_{35} = A_{12} A_{23}(A_{51} + A_{52}) + A_{15}(A_{21} A_{53} + A_{23} A_{52}).$$

We can neglect A_{15} in $(A_{12}+A_{15})$, $A_{23}A_{52}$ in $(A_{21}A_{53}+A_{23}A_{52})$ and A_{53} in $(A_{51}+A_{52}+A_{53})$. This gives

$$\frac{\mathfrak{A}_{35}}{\mathfrak{A}_{25}}=\frac{A_{12}A_{23}(A_{51}+A_{52})+A_{15}A_{21}A_{53}}{A_{12}A_{32}(A_{51}+A_{52})}.$$

Then

$$\alpha=\frac{A_{32}}{A_{23}}\times\frac{\mathfrak{A}_{35}}{\mathfrak{A}_{25}}=1+\frac{A_{15}A_{53}A_{21}}{A_{12}A_{23}(A_{51}+A_{52})}. \tag{3.1}$$

The weighted mean of the transition $2p$–$3s$, $2p$–$3d$ and $2s$–$3p$ does not differ appreciably from the value of α for $2p$–$3d$ alone. A result exactly similar to (3.1), namely,

$$\alpha=1+\frac{A_{15}A_{54}A_{21}}{A_{12}A_{24}(A_{51}+A_{52})}, \tag{3.11}$$

holds for H_β. Notice that in both cases α is of the form $\alpha=1+B/W$, where B is a constant.

The numerical results are $\alpha-1=0{\cdot}03$ for H_α and $\alpha-1=0{\cdot}17$ for H_β, both being calculated for $W=1/3$ and $T=6500°$. The low value of $\alpha-1$ for H_α is due to the low f-value of the transitions from $3d$ to the continuum.

So far we have supposed that there are no absorption lines, but owing to the presence of the absorption lines H_α and H_β the number of upward transitions $2\rightarrow3$ and $2\rightarrow4$ are diminished. The Lyman line is much more powerful and at the surface the transitions $1\rightarrow2$ are diminished, but it is one of the features of the formation of absorption lines that the stronger the coefficient of absorption the less deeply does the effect of the line penetrate, and the more complete is the banking up of radiation at small optical depths. This point is illustrated by a later equation (3.4), in which it must be noted that the constant q is very much greater for the Lyman line than for either H_α or H_β.

We consider first the conditions at a depth such that the Lyman radiation is sensibly restored to its equilibrium value, while H_α and H_β are unaffected. Let the intensities of these two lines be r_3 and r_4. The transition coefficients A_{23} and A_{24} will appear in the determinant multiplied by r_3 and r_4 respectively and we will get new values for α. From (3.1)

$$\alpha-1=\frac{1}{r_n}\frac{A_{5n}}{A_{2n}}\frac{A_{21}A_{15}}{A_{12}(A_{51}+A_{52})}, \tag{3.12}$$

where $n=3$ for H_α and $n=4$ for H_β.

In a well-known notation * the equation of transfer of radiation is

$$\cos\theta\frac{d\mathcal{J}_\nu'(\theta)}{\rho dr}=-(k+l)\mathcal{J}_\nu'(\theta)+k\mathcal{J}_T+\alpha l\mathcal{J}_\nu. \tag{3.2}$$

Suppose that $\mathcal{J}=\mathcal{J}_1$ when $0<\theta<\pi/2$ and $\mathcal{J}=\mathcal{J}_2$ when $\pi/2<\theta<\pi$. We ignore all variation of radiation intensity with direction, except for the broad distinction of inwards and outwards. We then have

$$\frac{d\mathcal{J}_1}{d\tau}=-(1+\eta)\mathcal{J}_1+\mathcal{J}_T+\alpha\eta\left(\frac{\mathcal{J}_1+\mathcal{J}_2}{2}\right).$$

* Eddington, *M.N.*, **89**, 622, 1929.

At the centre of the line η ($=l/k$) is large and we can neglect those terms which do not contain it; and at the surface, $\mathcal{J}_2=0$. We are left with

$$d\mathcal{J}_1/d\tau = -\eta(1-\alpha/2)\mathcal{J}_1,$$

so that $d\mathcal{J}_1/d\tau$ is positive if $\alpha > 2$. The absorption line is formed by the negative value of $d\mathcal{J}_1/d\tau$, so that $\alpha=2$ is a limiting value beyond which the absorption line will not form; and those values of r which give us $\alpha=2$ from (3.12) are lower limits to the central intensities of H_α and H_β. We find in fact

$$r_n = \frac{A_{5n}}{A_{2n}} \frac{A_{21}A_{15}}{A_{12}(A_{51}+A_{52})}, \tag{3.3}$$

so that lower limits to the central intensities of H_α and H_β are 0·03 and 0·17.

We must pause in our argument about the solar lines to emphasise a very important point which has application to the general question of the Rosseland cycle and the formation of reversals and bright lines in all spectra. Rosseland's theorem shows that in all cases where $W < 1$, then $\alpha > 1$. *But $\alpha > 1$ is not enough to form a bright line*—α must be greater than 2. The reason is simple enough. Take a cubic centimetre of gas above an atmosphere. It will remove a certain amount of energy from the outward flow: call this amount E. The material being fluorescent, the cubic centimetre will emit a greater amount, αE, where $\alpha > 1$. But only half of this is contributed to the outward flow—the rest goes back to the star. (We ignore all distinctions except inward and outward.) Unless $\alpha > 2$, then the outward radiation will gain nothing from our cubic centimetre, and there will be no contribution to a bright line.

This explains why it is that absorption lines can be formed at all, in a fluorescent region. Apart from the absorption line $\alpha - 1$ is only 0·17 even for H_β, and the line is free to deepen until $r=0{\cdot}17$ which makes $\alpha=2$.

The two lower limits for the intensities of H_α and H_β, $r=0{\cdot}03$ and $r=0{\cdot}17$, were found by ignoring the region in which the Lyman α line is below full strength, and this surface layer will tend to bring up the central intensities of both lines. We attempt a rough evaluation of the effect of the Lyman defection, but there are two physical quantities required in the evaluation (the relevant electron pressure and the range of frequency within which the fluorescent radiation appears) whose values cannot be stated precisely, so that the calculation is only concerned with the general order of magnitude of the brightening of the line to be expected as a result of the defection of L_α.

There is very little fluorescent radiation in L_α in comparison with the great number of transitions involved: we can put $\alpha=1$ in (3.2) and solve. The well-known solution is

$$\mathcal{J}_\nu = H(2+3\tau) - \frac{2H\eta}{1+\eta+2/3q}e^{-q\tau},$$

where $q^2=3(1+\eta)(1+\epsilon\eta)$. Near the surface the collision factor ϵ tends to zero, τ is small and η is large. We have

$$\mathcal{J}_\nu = 2H(1-\exp(-q\tau)), \tag{3.4}$$

so that

$$\mathcal{J}_\nu/2H = 1/r \qquad \text{when} \qquad q\tau = \log_e\left(\frac{1}{1-r}\right).$$

Let s_ν be the atomic coefficient of line absorption, equal to

$$\frac{\pi e^2}{m}\cdot\frac{f\delta}{(\nu-\nu_0)^2+\delta^2}$$

(in the absence of Doppler effect), where, on the classical theory, which is applicable to L_α,

$$\delta = \frac{\omega}{4\pi} = \frac{2\pi e^2\nu^2}{3mc^3}.$$

The Doppler effect will modify s_ν to s_ν', where *

$$s_\nu' = f\frac{\pi e^2}{m}\frac{\delta}{(\nu-\nu_0)^2}\frac{4}{\sqrt{\pi}}\int_0^d x^2e^{-x^2}dx + f\cdot\frac{\pi e^2}{m}\cdot\frac{c}{\nu}\left(\frac{nM}{2RT}\right)^{\frac{1}{2}}e^{-d^2},$$

where

$$d = c\left(\frac{\nu-\nu_0}{\nu}\right)\sqrt{\frac{M}{2RT}}.$$

Suppose there are n hydrogen atoms each of mass M per cubic centimetre of the atmosphere, which has a density ρ and a coefficient of continuous absorption k. Then $\eta = \frac{nM}{\rho}\frac{s_\nu'}{k}$. The solar atmosphere is taken to be a mixture of 61·5 hydrogen atoms to one metallic atom with a mean atomic weight 1·5, so that $\rho = 1{\cdot}5\times nM$. Let N be the number of hydrogen atoms above 1 sq. cm. at a depth x, then $dN = ndx$, and the optical depth τ will be given by $d\tau = k\rho dx$ giving $\tau = 1{\cdot}5\times kMN$. The condition $q\tau = \log_e\left(\frac{1}{1-r}\right)$ reduces to

$$\sqrt{2s_\nu'/k}\times 1{\cdot}5\times kMN = \log_e\left(\frac{1}{1-r}\right),$$

or

$$N = \frac{1}{1{\cdot}5\times M}\cdot\frac{1}{\sqrt{2}}\log_e\left(\frac{1}{1-r}\right)\times\sqrt{\frac{1}{ks_\nu'}}. \tag{3·5}$$

A little calculation is involved in finding the value of k. Let P be the electron pressure, p the gas pressure and $\bar{x}$ the ionisation (supposed constant). Then $k = \alpha\bar{x}PT^{-11/2}$, where α is a known constant. Since $\bar{x}$ is small, $P = \bar{x}p$. Now $dp = g\rho dx$ so that

$$dP = g\bar{x}\times 1{\cdot}5\times MdN.$$

Taking $\bar{x}$ constant

$$P = g\bar{x}\times 1{\cdot}5\times MN$$

and

$$k = \alpha' g\bar{x}\times 1{\cdot}5\times MN, \tag{3·6}$$

* *Ann. S.P.O.*, **3**, 83, 1933.

where $\alpha' = a\bar{x}T^{-11/2}$. We have to estimate $\bar{x}$: suppose that all the metallic atoms are once ionised and that about 1 per cent. of the hydrogen atoms are ionised: then $\bar{x} = 0{\cdot}026$. For the solar atmosphere $g = 2{\cdot}7 \times 10^4$ and $\alpha' = 0{\cdot}04$.* The value of s_ν' where $\nu = \nu_0$ is $2{\cdot}2 \times 10^9$. Eliminating k from (3.5) and (3.6) we find

$$N = \frac{1}{1{\cdot}5 \times M}\left\{\frac{\log_e (1-r)^{-2}}{2s_\nu' \alpha' g \bar{x}}\right\}^{1/3} = 1{\cdot}85 \times 10^{20}\left\{\log_e\left(\frac{1}{1-r}\right)\right\}^{2/3}. \tag{3.7}$$

Working with the data for $T = 6500°$ and $W = 1/3$ and putting r, s for the central intensities of L_α and H_α respectively, we get

$$\frac{1}{n_1}\{\tfrac{1}{2}n_3A_{32} - s \,.\, n_2A_{23}\} = 2{\cdot}61 \times 10^{-4} - 4 \times 10^{-6}s - 8{\cdot}86 \times 10^{-3}sr,$$

the quantity in the brackets being the gain, in quanta, to the emergent flux of H_α, due to 1 c.c. of gas.

When $r = 1$, $\frac{1}{2}n_3A_{32} - s \,.\, n_2A_{23} = 0$ when $s = 0{\cdot}03$, giving as before the central intensity of H_α when there is no Lyman defection. Using this value of s † we find

$$\frac{1}{n_1}\{\tfrac{1}{2}n_3A_{32} - s \,.\, n_2A_{23}\} = 2{\cdot}61 \times 10^{-4}(1-r).$$

Notice that

$$2{\cdot}61 \times 10^{-4} \equiv \tfrac{1}{2} \cdot \frac{A_{15}A_{53}}{(A_{51} + A_{52})}.$$

Now the total fluorescence emitted outwards by the layer in which $r < 1$ is

$$\int_0^1 dN \,.\, 2{\cdot}61 \times 10^{-4}(1-r).$$

N being given by (3.7) and being equal to $A\left\{\log_e\left(\frac{1}{1-r}\right)\right\}^{2/3}$. Then

$$dN = \frac{2A}{3}\left\{\log_e\left(\frac{1}{1-r}\right)\right\}^{-1/3}\frac{dr}{1-r}$$

and

$$\int_0^1 (1-r)dN = \tfrac{4}{3}A \times \int_0^\infty x^{1/3}e^{-x^2}dx = 0{\cdot}88 \times A.$$

Introducing numerical values we find for the emergent H_α radiation the value $0{\cdot}88 \times 1{\cdot}85 \times 10^{20} \times 2{\cdot}61 \times 10^{-4} = 4 \times 10^{16}$ quanta per sec. When we wish to compare this output in quanta per sec. with the brightness of the normal spectrum it is necessary to ask, in what range of frequency is it emitted? For the present we will suppose that the fluorescent emission

* *M.N.*, **93**, 693, 1933, from Chandrasekhar's formula. The adopted value of $\alpha'\bar{x}$ only enters our value of N in the one-third power.

† This value of s is appropriate to the boundary $r \to 1$, which is where the term involving s has most effect.

is distributed over a range of frequency equal to the half-width of the *emission* line, taking account of the Doppler effect. At the centre of the line $s_\nu' \propto e^{-d^2}$ where

$$\frac{d}{c} = \left(\frac{\nu - \nu_0}{\nu}\right)\sqrt{\frac{M}{2RT}},$$

we have for the half-width $d^2 = \log_e 2$ and $\frac{\nu - \nu_0}{\nu} = 2{\cdot}75 \times 10^{-5}$. The width $2(\nu - \nu_0)$ is equivalent to about 0·36 A. at H_α.

The radiation in this frequency—two-thirds of black-body emission at 6500°—is about 10^{10} quanta per sec. The fluorescent contribution to H_α from the upper layers is then about $r = 0{\cdot}20$, which, added to the radiation from below,* gives $r = 0{\cdot}23$.

The fluorescent contribution from the top layers is highly sensitive to the temperature of the radiation. For example, if we take 7000° instead of 6500° we find for H_α $r = 0{\cdot}63 + {\cdot}03 = 0{\cdot}66$. In the case of H_β the contribution from the top layer is much smaller than for H_α, because $n_4 A_{42}$ is much smaller than $n_3 A_{32}$ at these temperatures. The actual figures for the contribution from the top layer are 0·03 at 6500° and 0·13 at 7000°, making $r = 0{\cdot}20$ and $r = 0{\cdot}30$ in the two cases. H_β is accordingly much less sensitive to local conditions than H_α. This is, in fact, observed to be the case.

In the case of H_β the calculated fluorescent emission at 6500°, namely, $r = 0{\cdot}20$, only a small part of which rests on insecure data, shows a satisfactory agreement with the observed high central intensity of the line. Unsöld† gives about $r = 0{\cdot}25$. The same observer gives roughly the same central intensity for H_α. The present calculation suggests very roughly that the average central intensity of H_α should be about $r = 0{\cdot}23$, and that it should be highly dependent on local conditions—which, of course, the spectroheliograph shows it to be. It is admitted that the present calculation of the fluorescence of H_α rests on rather insecure data, but the order of magnitude of the phenomenon is amply sufficient to allow the opinion that fluorescence is the agent which is responsible for the high central intensities of the Balmer lines, and of the local bright patches in H_α spectroheliograms.

Mention should be made of the work of Professor Pannekoek,‡ who has recently investigated this problem and who came to the conclusion that the amount of fluorescence was insufficient to account for any substantial contribution to the observed high central intensities. Pannekoek considered Rosseland transitions in much the same way as we have done, but he did not take into account the Lyman transitions or transitions to the continuum. Our results differ from his, naturally enough.

* It is not supposed that this procedure is rigorously correct, since the upper contribution must affect the lower; but the general conclusions about H_α and H_β will not be affected by this difficulty. In the case of H_α nearly all the contribution comes from the upper layer, and in the case of H_β nearly all comes from the lower layer. Mutual influence of upper and lower layers will in both cases be small.

† *Zeitschrift f. Astrophys.*, **2**, 199, 1931.

‡ *B.A.N.*, **7**, 151, 1933.

§ 4. *The Appearance of the Solar Disc in the Spectrohelioscope*

Attention has recently been called by Chandrasekhar's theory of the chromosphere to the mottled appearance of the solar disc when photographed in integrated light. The disc shows the same mottled appearance in spectroheliograms and when examined in the spectrohelioscope. At first sight there is no immediate reason why the mottling in integrated light should be accompanied by a local increase in the central intensities of lines—which is what the spectrohelioscope shows. Ordinary local increase of temperature, giving a bright patch in integrated light, should merely increase the *absorption* intensity of the Balmer lines as a whole, having most effect in the wings of the line. Fluorescence, however, accounts for the phenomenon of mottling in the spectrohelioscope. According to the calculations of § 3 local variations in radiation temperature should have much more effect on the central intensity of H_α than on that of H_β—which is precisely what is observed. The mottling is very difficult to discern in H_β light.*

In addition to the mottling very bright patches are sometimes seen in H_α light. They are interpreted on the present view as being patches of hydrogen lying over a local area in which the emission of ultra-violet light is, for some reason, far above normal. They should be much more conspicuous in H_α than in H_β.

The prominences are exposed to dilute temperature radiation and might be expected to fluoresce strongly and to appear as bright markings above the solar disc, but, in fact, they appear dark. The reason for this, it is suggested, is as follows: The immediate surface layer of the Sun is responsible for brightening the central intensities of the Balmer lines, which it can only do by removing some of the quanta from the radiation beyond the head of the Lyman series. The radiation which reaches the prominences has come through the fluorescent layer and is accordingly a little deficient in ultra-violet light, which lessens the tendency of the prominences to fluoresce.† This division of the atmosphere into successive layers of absorption, fluorescent emission and feeble reabsorption is shown by the well-known triple structure of the K line of Ca^+. The last reversal K_3 is performed by thin clouds above the general level of the atmosphere.

Local bright patches are shown by spectroheliograms ‡ taken in wavelengths 4046, 4202 where there are strong *Fe* lines, and in 4078 (Sr^+). Accordingly, fluorescence is not confined to the hydrogen lines, though they show the phenomenon best, and are in the present state of affairs the only lines for which even the roughest calculation can be made.

There is undoubtedly some agency in the solar atmosphere which gives rise to bright reversals to all lines under exceptional conditions: it is here suggested that this agency is fluorescence, and calculations for H_α and H_β show that the effect of fluorescence is of the right order of magnitude to account for the observed phenomena in these two lines, namely, that the central intensity of H_α is variable and greatly affected by local conditions,

* d'Azambuja, *Ann. Obs. Paris*, **8**/2, 32, 1930.

† Poverty in ultra-violet light at the extreme surface must also impose a check on the H_α fluorescence.

‡ d'Azambuja, *Ann. Obs. Paris*, **8**/2, 57, 1930.

while that of H_β is much more steady and of amount about 20 per cent. of the intensity of the continuous background outside the line. It is further suggested that fluorescence is responsible for the central intensities of the Fraunhofer lines in general.

§ 5. *Bright-line Stellar Spectra*

Taking a fixed value of W—say $W = 1/2$—merely doubling the temperature from 6500° to 13,000° will not make $\alpha > 2$.* Bright-line emission is not an unescapable phenomenon in a B-type star. But increase of temperature aids enormously the process of fluorescent emission in a surface envelope, since the increase in temperature brings up the number of atoms in states 4 and 3. H_α will appear bright more easily than H_β because $n_3 A_{32}$ is greater than $n_4 A_{42}$.

Bright-line emission seems to be due to the presence of a sufficiently extended atmosphere which is so tenuous that the continuous absorption coefficient k is effectively zero. (If k is not zero there will be inflowing as well as outflowing radiation and the condition $W = 1/2$ is destroyed.) The usual form of the bright line in stellar spectra, namely, an emission line superposed on a very broad absorption line suggests an atmosphere surrounding a star which by itself shows the normal absorption line. In default of any theory of the origin and extent of such envelopes it seems a little artificial to make any calculations, especially as certain rather delicate questions (such as the strength of an absorption band at the head of the Lyman series in the original stellar spectrum) would enter into the results. Here again, however, there seems to be no doubt but that Rosseland's fluorescence argument is the explanation of bright-line emission, and here again the condition $\alpha > 2$ protects the Rosseland theory from predicting too much—bright lines for *every* B-type star.

ON THE EFFECT OF THE TOTAL LINE AND BAND ABSORPTION IN STELLAR SPECTRA.

G. Shajn.

The integral absorption due to the spectral lines and bands chiefly affects the observed colour index, the temperature and the absolute magnitude, as well as the spectrobolometric measurements, the phenomenon of the darkening to the limb and so on. The effect in question increases with the advancing type and plays a particularly important rôle for the variables of type M. The other side of this problem is the question of the subsequent adventures of the energy absorbed in the lines and bands. This was emphasized especially by Milne and Eddington.† The energy dammed

* The value of $\alpha - 1$ for H_α at $T = 13{,}000°$ and $W = 1/3$ is 0·12.

† *The Observatory*, **51**, 88, 1928; *M.N.*, **81**, 510, 1921; *Internal Constitution of the Stars*, p. 337.

back in the spectral lines has to be returned to the layers beneath and will be redistributed in some way over different wave-lengths. On the other hand, the energy returned to the photospheric layers keeps up a temperature higher than these latter would otherwise possess, thus producing such a kind of blanketing effect.

The present paper deals chiefly with the various effects of the total line and band absorption in the several problems mentioned above, leaving out of consideration the blanketing effect and other theoretical questions.

§ 1. As to the stars the observed data on the total line and band absorption are completely absent and for the Sun only this is suggested to be about 15 per cent. in the mean.* This number is based on the attempt by Minnaert to reduce the Rowland estimates R to the true absorptions i. Using an approximate relation between R and i found for longer wave-lengths and assuming black-body radiation Minnaert estimates this effect to be approximately equal to 15 per cent. Later it was found by Minnaert, van Assenbergh, Mulders and Woolley † that the relation between R and i depends to a considerable degree on the wave-length. For this reason 15 per cent. is an approximate estimate. In addition there are no data concerning the total line absorption for different regions along the spectrum.

Spectrograms of stars of different spectral types were obtained with the single-prism spectrograph attached to the 40-inch reflector at Simeis, giving the dispersion 36 A. per mm. near $H\gamma$. Ilford Monarch and Special Rapid Panchromatic plates were used. The spectrograms were standardized by means of a series of lamp spectra obtained with the same spectrograph, using for the most part a system of slits of different width. Some were standardized in analogous manner, using a wedge before the usual slit. Out of many spectrograms those giving the best photographic density for the self-registering microphotometer were selected. For different parts of the spectrum separate characteristic curves were drawn. At the dispersion used the spectral lines in late-type spectra are mostly blends. The fainter lines produce a kind of local depression in the continuous spectrum. It was found most reasonable and practical to measure all the visible details in the spectrum and, for the deeper lines, several points of their profiles. The measured intensities were then plotted on a graph and with a planimeter the total absorption was derived for the interval per 25 A. from λ 3750 to λ 4600 and per 50 A. from λ 4600 to λ 6550. The total line absorption is expressed in units of the intensity of the continuous spectrum.

The most serious error is caused by some uncertainty in the drawing of the continuous background. In several regions there is such a crowding of lines that the drawing of continuous background becomes very difficult. This is especially true for stars of type M in the region of molecular bands. However, as will be shown later, by selecting some points in the continuous spectrum probably free from bands and lines, and using Planck's curve, we are able to draw for several regions a more or less satisfactory theoretical line for the continuous background. Owing to a

* *Zs. f. Phys.*, 45, 610, 1927.

† *Zs. f. Phys.*, 53, 248, 1929; *Zs. f. Astroph.*, 1, 192, 1930; *M.N.*, 92, 806, 1932.

very large amount of measurements and computations every region was measured mostly on two different spectrograms and only in a few cases single measurements were made. The results of measurements for the stars under consideration, except α Lyræ and β Pegasi, are plotted on fig. 1.

The accuracy obtained is about 10 per cent. as derived from the internal agreement. However, systematic error is here of more importance. We must keep in mind that the dispersion used decreases from 19 A. per mm. near λ 3750 to 140 A. per mm. near λ 6500, and this may be connected with a changing systematic error along the spectrum. Besides, the drawing of the continuous background may depend on the region under consideration. To check the dispersion effect the total line absorption was measured in the region 4226–4260 of the solar spectrum obtained with two different spectrographs.

Single-prism Spectrograph	Three-prism Spectrograph
Dispersion 33·0 A. = 1 mm.	7·7 A. = 1 mm.
1st spectrogram $0{\cdot}26_1$	$0{\cdot}24_4$
2nd ,, $0{\cdot}23_9$	$0{\cdot}25_2$

At such an increase of the dispersion the above effect is negligible. This allows us to hope that at the ratio of dispersions 6 : 1 (at λ 3800 and λ 6500) the error in question is not too large, especially if the character of spectral lines does not change along the spectrum.

Some information about the error of our measurements may be also obtained in the following way. For separate regions of the solar spectrum per 100 A. we have computed from *Revision of Rowland's Preliminary Tables* the number of lines of different intensity. Using the correlation between R and i mentioned above,* we, strictly speaking, cannot find the total absorption for any interval since the correlation holds but for the moderate intensities and in addition it changes with the wave-length. However, for several regions of the spectrum the total absorption obtained in such a way is probably of value. The table below is self-explanatory.

The Total Line Absorption

	Present Paper (Measured)	Rowland's Estimates *via* Minnaert and van Assenbergh (Computed)
3950–4050	0·28	0·29
4200–4300	·23	·20
4450–4550	·13	·16
4700–4800	·09	·10
4950–5050	·12	·09

For the region 3950–5050 we have a fair agreement. In spite of the small dispersion, especially in the red, our measurements allow to bring out the colour equation of the Rowland scale. Thus comparing 4200–4800 with 5200–5800 we find the true absorption in the latter region to be approximately 1·6 times larger, the Rowland estimates being the same.

* *Zs. f. Phys.*, 53, 248, 1929.

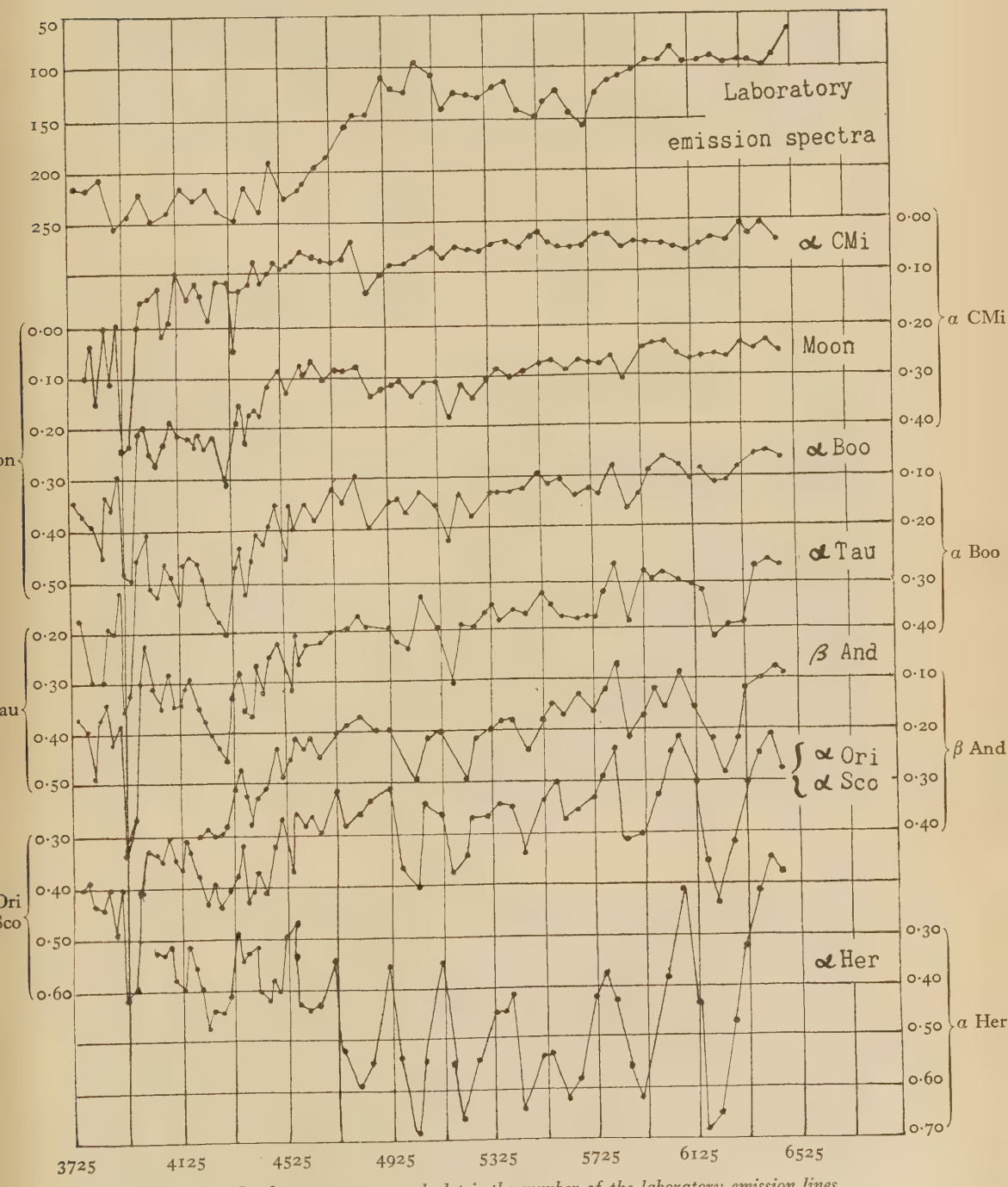


FIG. 1.—*In the upper curve each dot is the number of the laboratory emission lines per 50 A. In stellar spectra each dot is the total line and band absorption per 25 A.* (3750–4600) *and per 50 A.* (4600–6550).

This is near to the results of Minnaert and his collaborators.* If this colour equation extended into the violet we would expect the values of absorption based on our measurements to be smaller than the computed ones. But just in this region there are a lot of deep lines, and this should possibly lead to the decrease of the computed absorption. A comparison may be also made with regard to α Lyræ for which some other measurements are known.

Region	Absorption in Magn.
	m
3775–4975 (only hydrogen lines)	0.08_6 Günther †
3775–4975 (all the lines)	0.09_0 Writer
$H\delta - H\beta$ (only hydrogen lines)	0.05_6 E. G. Williams ‡
$H\delta - H\beta$ (,, ,,)	0.06_3 Günther
$H\delta - H\beta$ (all the lines)	0.07_2 Writer

The observations of E. G. Williams and Günther were reduced by the writer in a manner nearly analogous to the Simeis ones. It appears that our measurements of the total line absorption are not affected by a considerable systematic error, at least excluding the latter subdivisions of M and in general the red part of the spectrum for types Bo–Ko where the lines are faint. The accuracy of the measurements must not be overstressed, and our values of the total line absorption must be considered as preliminary ones.

§ 2. For a small region in the spectrum the measured total line and band absorption, as expressed in units of the intensity of the continuous spectrum for the same interval, is not influenced by the absorption in the atmosphere and in the instrument, or by the sensitivity of plate. But in studying the influence of the line and band absorption on the colour index and absolute magnitude, when all the visual or photographic region is considered, we must take into account the effect of the factors here mentioned, and also the distribution of the energy in the continuous spectrum. The observed intensity $\mathcal{J}_\lambda'$ turns out to be

$$\mathcal{J}_\lambda' = A_\lambda B_\lambda C_\lambda D_\lambda \mathcal{J}_\lambda, \qquad (1)$$

where $\mathcal{J}$ is Planck's function, and A, B, C and D are the factors depending on the absorption in the atmosphere and in the instrument, the plate or eye's sensibility, and the transmission of the filter (if used).

For our purpose there is no need to investigate these effects taken separately. The apparent distribution of the intensity in the continuous spectrum of α Lyræ (taken at 10° zenith distance) was studied and by the comparison with the black-body distribution of the energy ($T = 11{,}000°$) the factors $\alpha_\lambda = A \,.\, B \,.\, C$ were derived (the values of $\mathcal{J}_\lambda'$ were preliminarily reduced to the normal dispersion, multiplied by the factor $C/(\lambda - \lambda_0)^2$, where C and λ_0 are the constants of the Cornu-Hartmann formula.

As is evident from fig. 1, the total line absorption is far from being uniform along the spectrum, and this especially affects the colour index.

* *Loc. cit.* † *Zs. f. Astroph.*, 7, 106, 1933.
‡ *Annals of Sol. Phys. Obs. Camb.*, **2**, pt. 2, 1932.

Let the loss due to the total line absorption for an interval $d\lambda$ be k_λ. Then for the visual region from λ_1 to λ_2 the intensity will be equal to

$$\int_{\lambda_1}^{\lambda_2} \mathcal{J}_\lambda'(1-k_\lambda)d\lambda.$$

The loss in magnitude Δm for the region under consideration is

$$\Delta m = -2{\cdot}512 \log \left\{ \frac{\int_{\lambda_1}^{\lambda_2} \mathcal{J}_\lambda' d\lambda}{\int_{\lambda_1}^{\lambda_2} \mathcal{J}_\lambda'(1-k_\lambda)d\lambda} \right\}. \qquad (2)$$

We shall also have an analogous expression for the photographic region from λ_3 to λ_4. Then the change in colour index Δ (C.I.) is

$$\Delta\,(\text{C.I.}) = 2{\cdot}512 \log \left\{ \frac{\int_{\lambda_3}^{\lambda_4} \mathcal{J}_\lambda' d\lambda}{\int_{\lambda_1}^{\lambda_2} \mathcal{J}_\lambda' d\lambda} \cdot \frac{\int_{\lambda_1}^{\lambda_2} \mathcal{J}_\lambda'(1-k_\lambda)d\lambda}{\int_{\lambda_3}^{\lambda_4} \mathcal{J}_\lambda'(1-k_\lambda)d\lambda} \right\}. \qquad (3)$$

Using k_λ from fig. 1 and the observed or computed values of $\mathcal{J}'$ we easily find Δ (C.I.).

The derived values of Δm and Δ (C.I.) are given in Table I [using $\alpha\lambda$ for Ilford Monarch plates (photographic region) and Ilford Special Panchromatic plates with Jena filter OG_1 (visual region)]. In the latter case the transmission of the filter was taken into account. The optical absorption was assumed to be equivalent to that for our spectrograph. For this reason the data of Table I are of special interest. For a given case it is necessary to find $\alpha\lambda$ and then to compute Δm and Δ (C.I.) using the absorptions represented in fig. 1. The data of Table I, however, are probably applicable to the assumed systems of the photographic and photovisual magnitudes. As it will be shown later, the line and band absorption in the visual region for α Herculis is possibly to be considered rather as an inferior limit.

TABLE I

	Photographic Region		Visual Region		
	Loss in Units of the Intensity of continuous Background	Loss in Magn.	Loss in Units of the Intensity of continuous Background	Loss in Magn.	Δ (C.I.)
		m		m	m
α Lyræ Ao	0·07$_9$	0·09	0·00$_6$	0·00$_7$	−0·08$_3$
α Can Min dF5	·13$_9$	·16$_2$	·04$_7$	·05$_2$	−0·11$_0$
Moon dGo	·19$_6$	·23$_6$	·08$_6$	·09$_8$	−0·13$_8$
α Bootis gKo	·24$_2$	·30	·11$_5$	·13$_3$	−0·16$_7$
α Tauri gK5	·28$_9$	·37	·14$_4$	·16$_9$	−0·20$_1$
β Androm gMo	·28	·35$_7$	·16$_8$	·20$_0$	−0·15$_7$
α Orionis cM2	·33$_6$	·44$_5$	·25$_1$	·31$_4$	−0·13$_1$
α Scorpii cM2	·32$_3$	·42$_4$	·23$_7$	·29$_4$	−0·13$_0$
β Pegasi gM2	·28$_8$	·36$_9$	·23$_4$	·29$_0$	−0·07$_9$
α Herculis gM5	·41$_6$	·58$_4$	·47$_5$	·70$_0$	+0·11$_6$

While the absorption of individual lines considerably increases with the advancing type, the small change of the total line and band absorption in the photographic region from F5 to M2 is accounted for by the decreasing rôle of the far violet region (very rich in deep lines) with the decreasing temperature owing to the displacement of the maximum intensity of the

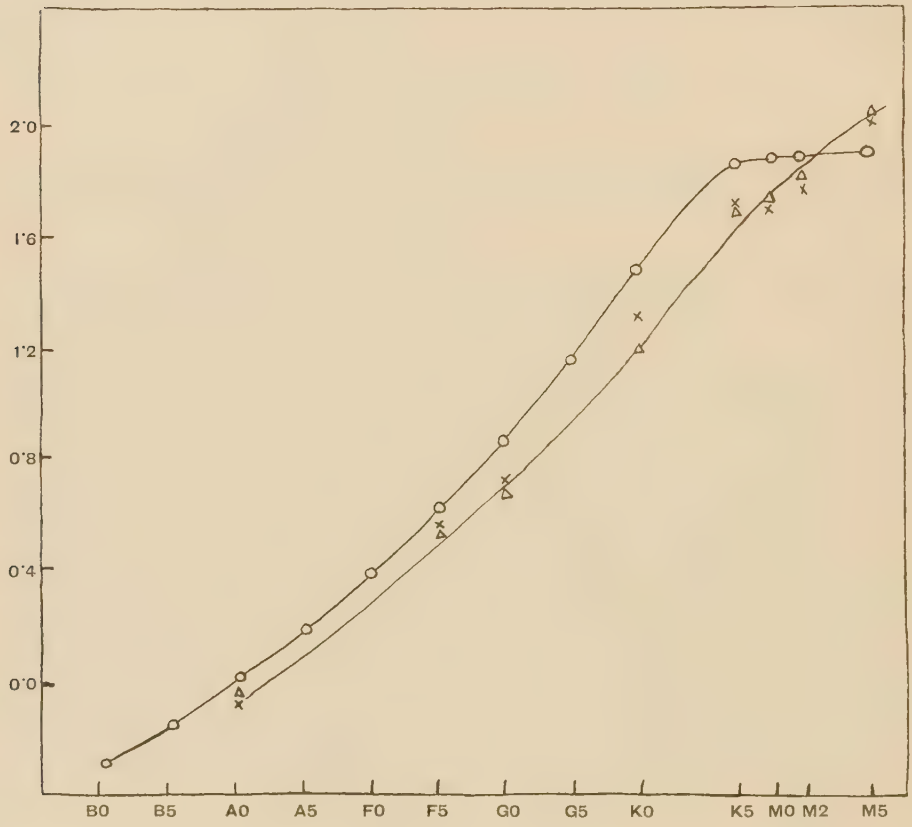


Fig. 2.—*Spectrum and Colour Index.*

O = *observed C.I.*
× = *C.I. corrected for the line and band absorption.*
△ = *computed C.I. (black-body radiation).*

continuous spectrum to the red. As to M stars the effect of the bands *TiO* becomes considerable even in the photographic region.

§ 3. Though the colour index depends on many factors, the correlation between C.I. and temperature is well established. Some uncertainty in this relation holds for the early stars of type B and the later subdivisions of type M. The cause of this is probably different for the two cases.* We may assert with certainty that C.I. from K5 to M5 is constant or even

* The slight delay in the increase of C.I. from B8 to F5 in several systems of C.I. is probably due, as is suggested by Öhman (*Ups. Medd.*, 48, 1930) and especially by W. Becker (*Zs. f. Astroph.*, 7, 281, 1933), to the influence of the total absorption effect of hydrogen lines.

decreases. It is evident that the observed C.I. are affected by the lines and bands. In this respect Table II and fig. 2 are instructive.

TABLE II

Spectrum	T	C.I. (observed) Mt. Wilson Syst.	C.I. (observed) Mt. Wilson Syst. Corrected for the Total Line and Band Absorption	C.I. (computed) Formula (5)	C. – O.
Bo	23000 (I)	−0·32			
B5	15000 (I)	−0·17			
Ao	11000 (I, II, III)	0·00	−0·08	$+0·00_5$	$+0·08_5$
A5	8600 (I, II)	0·19			
Fo	7400 (I)	0·38			
F5	6500 (I, II)	0·62	+0·51	+0·50	−0·01
Go	5600 (I, II, IV)	0·86	+0·72	+0·70	−0·02
G5	4700 (I)	1·15			
Ko	4200 (I, III, IV)	1·48	+1·31	+1·16	−0·15
K5	3400 (I, II, III)	1·84	+1·64	$+1·60_5$	$-0·03_5$
M2	3100 (I, II, III, IV)	1·88 (Ma = Mo − M2)	+1·76	+1·83	+0·07
M5	2850 (I, II, III, IV)	1·88 (Mb = M5)	+2·00	$+2·05_5$	$+0·05_5$

The temperatures are taken from *Astronomy*, Russell, Dugan and Stewart, Table XXX (I—from colour index, II—from energy distribution, III—from diameter determination, IV—from heat index). The corrections for the fourth column are taken from Table I.

In the case of black-body radiation C.I. and T are known to be connected by a formula of the type

$$\text{C.I.} = \frac{1·5548}{T}\left(\frac{1}{\lambda_{\text{eff (phot)}}} - \frac{1}{\lambda_{\text{eff (vis)}}}\right) + \text{const.} = \frac{A}{T} + B. \qquad (4)$$

For well-known reasons this formula is not very accurate.

Using the data from Table II we find

$$\text{C.I.} = \frac{7867}{T} - 0·71. \qquad (5)$$

The agreement between C. and O. is satisfactory though the residuals bring out a systematic error. A better agreement could be hardly expected if we keep in mind that the corrections and the observed C.I. do not belong to the same system. We can now assert that the constancy or even the decrease of the colour index from K5 to M5 is accounted for by the total line and band absorption effect.

Bottlinger has found a sensible decrease of C.I. from K5 to Md from his photo-electric observations. Using the data from fig. 1 and from Bottlinger's fig. 5 * we find the effect in question to be relatively larger for the photo-electric C.I. than for the usual ones.

While $d(\text{C.I.})/dT$ depends to a large extent on the constants of the formula

* *Handbuch der Astroph.*, **2**, 361, 1929.

(5) and therefore on the system of C.I. used, the correction for the line and band absorption depends on this system to much smaller degree. The smaller is the colour equation [C.I. (K0–A0)] the more will the correlation between C.I. and *T* or spectrum be relatively affected by the total line and band absorption. When computing the effect in question for Bottlinger's system of C.I. we find Δ (C.I.) for α Tauri and α Herculis to be $-0^m.13$ and $+0^m.18$ respectively. Therefore the differential effect here is nearly the same as for the Mount Wilson system ($0^m.31$ and $0^m.31_7$), while the colour equations are $0^m.55$ and $1^m.48$ respectively. This accounts for the much larger decrease of Bottlinger's C.I. for later subdivisions of M as compared with the Mount Wilson ones. When applying the above-mentioned corrections to Bottlinger's C.I. we find that the formula of type (4) holds also for this system. Generally, the different systems of C.I., when corrected for the line and band absorption effect, become more accordant, even in details.

This discussion shows that any conclusion with respect to the deviation from the black-body radiation of the stars may be drawn only when the account is taken for the line and band absorption effect. Unfortunately the stars of Table II are not homogeneous as to their luminosity. But this cannot seriously affect the results.

§ 4. It is known that the individual stars of the same spectral class and luminosity sometimes exhibit considerable differences in the strength of spectral lines. For instance, α Arietis (K2) and γ Aquilæ (K2) nearly of the same absolute magnitude (+0·7 and +0·1 respectively) show such a difference in the strength of lines. The measured total line absorption for the region 4200–4425 show that this is 1·41 ± 0·05 times larger for γ Aquilæ. Assuming that the mean ratio 1·41 holds along the whole spectrum, we can with some approximation estimate the effect in question :

	Loss in Magn.		
	Photogr.	Photovis.	Δ (C.I.)
	m	m	m
α Arietis . . .	0·34	0·15	−0·19
γ Aquilæ . . .	0·51	0·22	−0·29

Therefore, the stars with stronger spectral lines must bring out some colour excess, unless this is compensated by the change in the redistribution of the intensity of continuous spectrum.

As to the effect of the strengthening of lines depending on the luminosity, our material is too scanty for any conclusion. But there is no doubt that the supergiants α Orionis and α Scorpii show a considerable strengthening of lines. In another measured supergiant γ Cygni (cF8) the total line absorption is almost the same as that for α Arietis (K2). A small luminosity effect for the bands *TiO* may be also suggested. Therefore, when discussing the dispersion of the colour indices within a spectral class, the effect due to the strengthening of spectral lines must be taken into account.

§ 5. The effect in question is also to be mentioned in connection with the study of the extra-galactic nebulæ. It is well known that the large displacements of the wave-lengths in the spectra of these objects produce a

sensible change in the colour index, in the photographic and photo-visual magnitude (Hubble).* This is a result of the redistribution of the energy in the continuous spectrum. However, an additional effect in the nebulæ due to the influence of absorption lines is to be indicated. With large radial velocities the ultra-violet and the far violet part of the spectrum (rich in deep lines) are displaced into the usual photographic region, while the visual region is compensated in this respect but little (see fig. 1). For an accurate computation of the effect in question there are no observed data as to the total line absorption in the region 3400–3700. However, laboratory data and general impressions show that in Go-type stars the line absorption in this region is not smaller than in the neighbouring one, where according to our measurements it is equal to about 35 per cent. Assuming this absorption for the ultra-violet, we find for a Go-type star as follows :—

	Δ (C.I.)
Effect due to the total line absorption at the velocity $v = 0$ km./sec. .	-0.13_8
Doppler effect for $v = 20000$ km./sec.	-0.10
Doppler effect for $v = 20000$ km./sec. *plus* the effect due to the total line absorption	-0.30_2

In these computations the figures given by Stobbe were useful.†

Using the data from fig. 1 we are able to compute the effect in question for any other velocity. The correction in C.I. for this effect is usually larger than that due only to the Doppler effect.

§ 6. The temperature determinations based on the study of the energy distribution may be affected by the spectral lines. This is especially true if the spectra are obtained at low or moderate dispersion. Particularly the results are affected by the considerable depression in the violet in stars of later types. But in A-type stars the depression also becomes very sensible owing to the clustering of the lines of Balmer series, as was pointed out by Miss Payne.‡ The determination of the temperature gradient

$$G = 0.921 \frac{d\Delta m}{d(1/\lambda)} = \text{const.} - \frac{c_2}{T}$$

may be influenced with respect to this effect by errors of systematical and accidental character. This is connected, on the one hand, with the increase of the total line absorption in 3750–4450, and, on the other, with the non-uniform distribution of the total line absorption within this region. For instance, in α Tauri the mean total absorption in 3800–3950 is nearly 46 per cent., while for the region 6300–6500 it is about 7·5 per cent. On the other hand, within the limits of the regions in question we find large variations of accidental character. When a low dispersion is used the measured temperature for late-type stars will be smaller than the true one. In addition, this systematic error will probably increase with advancing type. To diminish this error the dwarfs of type Go are to be recommended as standards in the temperature determinations of late-type stars. Of course

* *P.A.S.P.*, **41**, 247, 1929.
† *A.N.*, **243**, 53, 1931.
‡ *Harvard Reprint*, No. 46, 1928.

there are also other reasons for selecting these stars as standards. It is true there is a depression in the violet in the spectra of type A due to the Balmer lines, but its character considerably differs from that in late-type stars. The error in the temperature determinations due to the depression in the violet may reach hundreds of degrees if the observed intensities of different points of the continuous spectrum are not previously corrected for the effect in question. Such corrections may be derived from fig. 1. Correcting in such a manner the observed energy distribution in the continuous spectrum (for instance, see Brill's discussion) * we are able to account to a large extent for the depression in the violet, which at first glance seems to indicate a serious deviation from the black-body radiation. This question is considered in detail below when reducing Abbot's spectrobolometric observations. The influence of the effect under consideration on the colorimetric determinations of temperature is more complicated, and for hydrogen lines this is investigated by Fessenkoff.†

§ 7. As is known, Abbot's spectrobolometric observations of the energy distribution in the solar spectrum bring out a sensible deviation from Planck's curve, the shorter wave-lengths being less intense than is to be expected theoretically. ‡ The same effect is also shown by the spectrophotometric observations of Wilsing. § On the other hand, there is a considerable difference between Plaskett's and Abbot's curves; || the former is based on spectrophotometric measurements, special attention being given to the selection of points in the continuous spectrum free from spectral lines. Plaskett's result, which is in agreement with black-body radiation, leads to a higher temperature. An agreement with black-body radiation is shown by the observations in the ultra-violet by Fabry and Buisson; ¶ here care is also taken to make the measurements free from spectral lines. It was suggested that the cause of this difference is the total line absorption affecting the results of Abbot and Wilsing.

Strictly speaking, it is impossible to correct the observations under consideration using our values of the total absorption derived from fig. 1. But taking the mean for a small region near the wave-lengths indicated by the observers we are probably able to some degree to account for the effect in question. On fig. 3 the dotted curves represent the black-body radiation for 6300° and 6700°; Plaskett's observations are represented by the symbol ⧫ (the vertical lines which pass through each observed point being determined in length by the limiting probable error); Abbot's observations are represented by ϕ, the vertical lines which pass through the circles being determined in length by the difference between the series 1903–10 and 1916–18. The crosses (×) represent Abbot's observations corrected in the manner mentioned above. Fig. 3 is nearly identical with fig. 9 in

* *A.N.*, **219**, 353, 1923.
† *Bull. Acad. Science Leningrad*, No. 6, p. 787, 1931.
‡ *Annals Astroph. Obs. Smith. Inst.*, **3**, 197, 1913, and **4**, 194, 1922.
§ *Publ. Astroph. Obs. Potsdam*, Bd. 22, No. 66, 1913.
|| *Publ. Dom. Astroph. Obs.*, **2**, No. 12, 251, 1923.
¶ *C.R.*, **175**, 156, 1922.

Plaskett's paper. Having in view the errors of observation of Plaskett and Abbot, and the approximate character of our corrections, we may conclude that the disagreement has practically disappeared. However, this must be considered as an illustration rather than as evidence. Small corrections to Abbot's measurements to reduce to the centre of the Sun and to make them strictly comparable with Plaskett's observations do not change the results. The recent observations by Pettit,* when corrected in such a manner, probably still show a deviation from the black-body radiation in the violet.

§ 8. As to the interpretation of the correlation between the spectrum and the luminosity it is essential to know the numerical value of the light loss in photographic and visual magnitudes due to the total absorption in

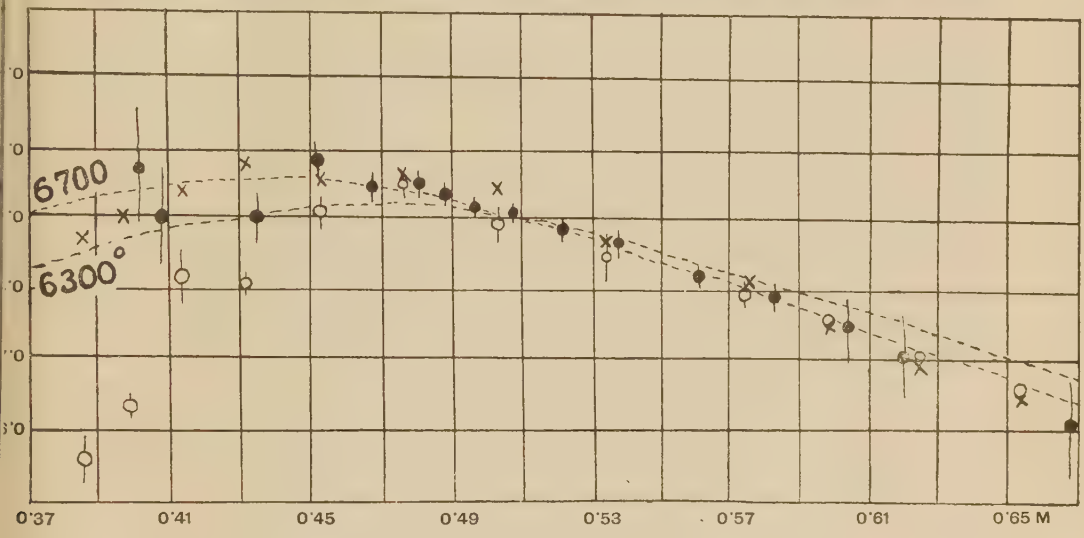


FIG. 3.— ● = *H. H. Plaskett (spectrophotometric).*
○ = *Abbot (spectrobolometric).*
× = *Abbot (corrected for the effect of the absorption lines).*

the spectral lines and bands. This will permit the separation in some problems of the *real* luminosity effect from the *apparent* one due to the lines and bands. For instance, such problems are as follows : (1) Spectral subdivision in M-type stars as criterion of absolute magnitude ; (2) the correlation between the spectrum or colour, the period and absolute magnitude in long-period variables ; and (3) the correlation between the amplitudes of the long-period variables in the bolometric, visual and photographic scale. Some problems of the long-period variables were already discussed by Miss Payne and ten Bruggencate in a somewhat different aspect than in this paper. †

The loss in question is represented in Table I. For the reasons mentioned above, the loss increases slowly to Mo. Owing to the uncertainty of the continuous spectrum the total line and band absorption in the later subdivisions of type M cannot be measured with the same accuracy as that

* *Ap. J.*, 75, 185, 1932. † *H.B.*, No. 876, 1, 1930.

for other spectral types. It is worth while to mention here the procedure of deriving the total absorption in M-type stars. Using the laboratory data with respect to the spectrum of *TiO* (Kayser's *Handbuch der Spektroskopiei*) we were able to find some small regions almost free from the lines and bands, or affected by them but slightly. For instance, between 6276 and 6357 there are only two faint bands. In this respect fig. 4 is instructive.

Using Planck's curve and the data with respect to the constants of the plate and of our optical system we were able to draw the theoretical line of the continuous spectrum, adjusting this to the observed one near λ 6300. In the spectrum of type M2, where the bands of *TiO* are of moderate strength, the theoretical line of the continuous background for $T = 3000°$ coincides with that usually drawn through the points apparently free from the lines and bands. For the stars of type M5 only the computed continuous background can be used. For stars later than M5 even this method becomes probably doubtful. Some objections may be raised against the procedure assumed, but it appears to be the best we can do in this case. Incidentally, the impression is that the bands in the red are less strong than in the green and yellow, the laboratory intensity being the same.

If the appearance of the bands is a single temperature effect, the *apparent* luminosity effect due to the bands must be the same for the giants and dwarfs. But this is not so. The few observations taken by us do not at present allow us to assert a dependence of the strength of the bands on the luminosity within the limits of giants. Table III is instructive with respect to the effect in question.

TABLE III

Absolute Magnitudes of the Stars of Type M

Long-period Variables		Giants		Dwarfs
(Merrill and Strömberg *)	Gerasimovič †	(Adams, Joy, Humason ‡)		
M2e −0·6	−2·3	M0	−0·1	+ 8·3
M3e −0·6		M1	0·0	8·8
M4e −0·5		M2	−0·2	9·3
M5e 0·0	−1·1	M3	−0·2	9·8
M6e +0·5		M4	−0·4	10·5
M7e +0·5		M5	−0·2	11·4
M8e +0·6	+0·3	M6	−0·4	12·5

The absolute magnitudes of Me stars were derived from the proper motions, and for well-known reasons the accuracy must not be overstressed. As to the stars of type M spectroscopic absolute magnitudes are available. The reduction curves for the dwarfs were based on the trigonometric parallaxes, and for the giants they were obtained for each spectral subdivision separately with the aid of mean parallaxes derived from peculiar

* *Ap. J.*, 59, 107, 1924.
† *H.R.*, No. 54, 1928.
‡ *Ap. J.*, 64, 225, 1926.

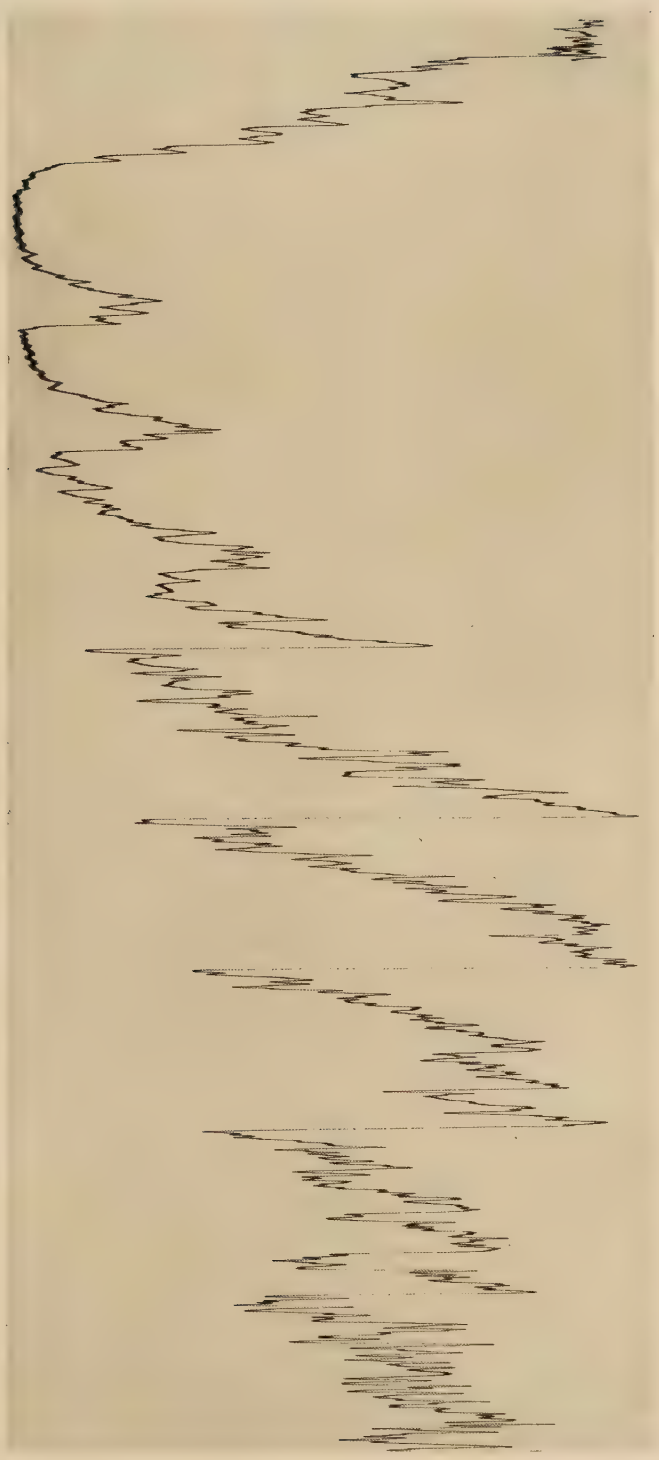


Fig. 4.—Microphotogram of α Herculis (visual region).

and parallactic motions.* In deriving the fifth column of Table III four supergiants of type Mo and three of type M1 and M2 were excluded.

As to the differential effect we have as follows: (1) M8e–M2e = +1·1 and +2·6; (2) gM6–gMo = −0·3; and (3) dM6–dMo = +4·2. Of course, the superposition of the *real* and *apparent* luminosity effects seems to be possible. No doubt for the dwarfs there is a large *real* luminosity effect. Our direct measurements (Table I) account wholly for the luminosity effect as derived by Merrill and Strömberg, and only for somewhat more than one-half of that derived by Gerasimovič. On the other hand, the giants of type M do not evidently clearly exhibit an apparent luminosity effect.

§ 9. We shall now show that the value of the line and band effect may be derived with some approximation immediately from radiometric measurements. As was mentioned above, the energy absorbed in the lines and bands must be re-emitted in different frequencies, so that the bolometric magnitude remains unchanged. The bolometric magnitude may be derived from radiometric measurements when account is taken of the absorption in the Earth's atmosphere and in the instrument, and in this case it is affected by the line and band absorption probably but little. On the other hand, the bolometric magnitude may be obtained from the visual absolute one, subtracting the bolometric correction:

$$M_{vis} - M_{bol} = \frac{29500}{T} + 10 \log T_e - 42{\cdot}53. \qquad (6)$$

The accuracy of this formula must not be overstressed for well-known reasons. It is evident that the bolometric magnitude derived in this manner is affected by the line and band absorption to the same degree as M_{vis}, unless the absorbed energy is re-emitted again in the visual region. It is possible that the bolometric correction itself is not totally free from the effect in question, since the derived temperature is influenced by lines and bands producing some redistribution of the energy in the continuous spectrum.

Generally we may write

$$M_{bol} - M_{vis} + \Delta M_{bol} = \Delta m_a \qquad (7)$$

where ΔM_{bol} is the bolometric correction and Δm_a the loss on M_{vis} due to the line and band absorption. On the other hand, for M_{bol} derived from the radiometric measurements we have according to Pettit and Nicholson † the following approximate formula:

$$M_{bol} = m_r - \Delta m_r + 5 \log \pi + 5{\cdot}9 \qquad (8)$$

* When using the trigonometric parallaxes for these stars (excepting supergiants) from Schlesinger's Catalogue we find according to formula (*Ap. J.*, 59, 97, 1924)

$$M = 5 + 5(\log \Sigma\pi - \log \Sigma 10^{-0{\cdot}2m})$$

the mean absolute magnitudes as follows: −0·2 for Mo–M1 (30), −0·5 for M2–M3 (25) and −0·1 for M4–M6 (10 stars).

† *Ap. J.*, 58, 302, 1928.

where m_r is the apparent radiometric magnitude and Δm_r is the correction for absorption in the Earth's atmosphere. The additional constant 0·9 relates to the adjustment of the radiometric and bolometric scales. Comparing (7) and (8) we are able to derive Δm_a. In such a manner Table IV is computed. (M_{vis} is compiled by van Maanen from all the available sources.)

TABLE IV

Type n	$M_{bol} - M_{vis}$		Δm_a			
			a	*b*	*c*	*d*
gK5 (3)	−1·2	± 0·1₄	+0·14	−0·04	−0·15	−0·19
dK5 (1)	−0·8					
gMo (1)	−1·5		−0·02	−0·20	−0·20 (1)	−0·20
dMo (3)	−0·9	± 0·0₇				
gM1–2 (8)	−1·7₄	± 0·2₃	+0·05	−0·13		−0·11
dM1–2 (2)	−1·8	± 0·0			−0·28 (3)	
gM3 (5)	−2·1₄	± 0·1₄	−0·07	−0·25		−0·30
gM4 (5)	−2·7	± 0·1₂	−0·44	−0·62		−0·35
gM5 (4)	−3·4	± 0·4₆	−0·90	−1·08	−0·70 (1)	−0·60
gM6 (4)	−4·1₅	± 0·4₄	−1·55	−1·73		−1·35
gM7 (3)	−4·0₇	± 0·2₂	−1·36	−1·54		−1·03
gM8 (3)	−4·6	± 0·5₂	−1·63	−1·81		−1·44
Me max. (10)	−4·1₃	± 0·5₃	−0·21	−0·39		−1·07
Me min. (10)	−·90	± 0·7₆	−1·48	−1·66		−2·67

In the column *a* the total line and band absorption is given as it is immediately obtained from (7) and (8). M_{bol} and M_{vis} are taken from Pettit and Nicholson's paper. ΔM_{bol} is computed according to (6). In the column *c* are given the results of our direct observations (Table II). The zero point of M_{bol} and ΔM_{bol} must not be stressed here; thus, adjusting this to bring Δm_a for Mo in the coincidence with the observed value under *c*, we obtain the column *b*. The columns *a* and *b* are referred to the temperature system as it is given in Russell, Dugan and Stewart (Table XXX). In column *d* Δm_a are referred to the temperature system derived from Table III of Pettit and Nicholson's paper * (water-cell absorption temperatures). These temperatures are lower than the adopted ones. The adjusting of the zero-point in this case too is arbitrary, but it is of interest to know the differential effect. We cannot be sure that the line and band absorption is the sole cause of the residuals Δm_a in Table IV, but the general agreement with direct measurements and other considerations lead to the suggestion that this is probably so.

It is very important to note that this method leads to the conclusion that the maximum absorption due to the bands (M8 minus K5) does not surpass $1^m.8$. Perhaps it is also of interest to note the rapid increase of Δm_a from M2 to M5 (see also the direct measurements) and the very slow increase from M6 to M8. The last perhaps is an indication of the approach to the maximum of the concentration of the molecules of *TiO* (the maximum near 2000° is also suggested by the equation of dissociative equilibrium of

* *Loc. cit.*, *Ap. J.*, 58, 302, 1928.

the diatomic molecules *TiO*).* If the band absorption might be accurately measured for all the subdivisions of M and the later ones of K with a spectrograph with higher dispersion than that used by us, the method in question would, on the contrary, permit the derivation of the bolometric correction and therefore the temperature. In addition, the temperature determinations of the stars of type M based on colour indices (corrected for the line and band absorption) seem to be available. The temperature scale determined in such a manner would be interesting since the usual methods give very discordant results or fail completely.

§ 10. The effect in question is to be mentioned in connection with the computations of angular and linear diameters for the later type stars. The usual formula

$$\log d'' = \frac{5900}{T} - 0{\cdot}20 m_{\text{vis}} - 3{\cdot}05, \qquad (9)$$

although not very accurate, shows a good agreement with the direct interferometer measurements to M2 inclusively (Russell, Dugan and Stewart, Table XXIX). It is evident that the band and line absorption is to be taken into account here for the later subdivisions of M. Table XXIX of Russell, Dugan and Stewart is extended as follows :—

	d'' (Measured)	d'' (Computed without Correction for Band Absorption)	d'' (Computed with Correction for the Band Absorption as found in this Paper)	d'' (Computed with Correction for the Band Absorption as suggested in *Ap. J.*, 63, 331)
	"	"	"	"
α Hercul.	0·030	0·023	0·031	0·050
o Ceti (max.)	0·056	0·045	0·056	0·107

The results depend to a certain degree on the assumed values of T (2800° for α Herc. and 2400° for o Ceti are values very near to those given in Russell, Dugan and Stewart, Table XXX). For o Ceti m_{vis} was assumed to be equal to 3·8 (epoch of Pease's measurements). When correcting for the total absorption, $-0^{m}{\cdot}7$ for α Herc. and $-0^{m}{\cdot}8$ (extrapolation) for o Ceti, we find a better agreement. The corrections suggested by Joy are $-1^{m}{\cdot}9$ for o Ceti and $-1^{m}{\cdot}7$ (extrapolation) for α Herc. Changing T within the reasonable limits we find that our corrections represent the observed values of the diameters better. The significance of this result must not be overestimated, but it is in good agreement with the moderate size of the corrections. The same follows if one computes T from (9), taking d'' from the interferometer determinations.

§ 11. The *apparent* luminosity effect due to the bands may be considered as a source of trouble in the spectroscopic determinations of the absolute magnitudes of M-type stars. This is partly connected with the fact that the total band absorption affects primarily the visual region of the spectrum, while the spectroscopic luminosity determinations are based on the photographic one. In any case the *real* luminosity effect will be confused with the *apparent* one for the visual region. In order to avoid this defect the

* Y. T. Cambresier and L. Rosenfeld, *M.N.*, **93**, 710, 1933; R. Richardson, *Ap. J.*, **78**, 354, 1933.

reduction curves must be derived for each spectral subdivision separately. In the early work this precaution has not been observed. It seems possible that the large errors of the absolute magnitudes of M-type stars determined spectroscopically are partly due to the effect in question.

§ 12. One of the consequences of the band absorption effect should be the correlation between the spectrum, period and absolute magnitudes in the long-period variables. Long ago a correlation between the spectrum and period was found. Recently, as was mentioned in § 8, Merrill and Strömberg found a correlation between the spectrum and luminosity. On the other hand, Gerasimovič derived a correlation between the period and luminosity: $M = -2{\cdot}3$ for 90–250, $M = -1{\cdot}1$ for 251–340 and $M = +0{\cdot}3$ for $P > 340$ days.* According to Gyllenberg the effect in question is still larger.† From Gyllenberg's discussion we are able to estimate the mean spectrum of three groups under consideration: M2e, M5e and M8e. There arises an important question whether the period-luminosity relation is due either to the *real* or *apparent* luminosity effect. Our direct measurements give $0^m{\cdot}7$ for the loss due to the line and band absorption for M5 in the photovisual scale. This amounts approximately to $0^m{\cdot}8$ in the visual one. Other considerations also lead to a value of the same order (about 1^m). This is sufficient to account for the spectrum-luminosity relation found by Merrill and Strömberg and rather more than one-half of that found by Gerasimovič (first two groups). It is true that our direct measurements do not extend farther than M5. The considerations discussed in § 9 account for about two-thirds of Gerasimovič's range between M2e and M8e. If we take Joy's approximate value of the band absorption ($1^m{\cdot}9$ from K5 to M6, as well as from M6 to M9), we obtain a sufficient amount of absorption to explain totally the relation under consideration.‡ It is possible that the strength of bands in visual giants and in Me stars is different, the spectrum being the same.

The discordant results with respect to the period-luminosity relation in galactic clouds when using the photographic magnitudes of the long-period variables is also rather an argument in favour of the moderate size of the effect under consideration.

A crucial test may be the new derivation of the absolute magnitudes of giants of type M (without characteristics *e* and apparently non-variables). The existing data appear not to show a considerable luminosity effect with the advancing type (§ 8).

§ 13. As is well known, in Me stars the visual amplitudes are much larger than the radiometric or bolometric ones. The variation in the brightness is connected here with the variation in the spectrum and temperature. Since the radiometric measurements are probably affected but slightly by the bands, the increase of the strength of the bands in the minimum seems to account partly for the large difference in the amplitudes

* *H.R.*, No. 54, 1928. † *Medd. Lund. Observ.*, ser. ii, No. 53, 1929.

‡ It is to be mentioned that the high increase of the solar velocity and of the mean peculiar velocity (from 18 to 83 and from 21 to 56 km./sec. respectively) with the decreasing period is an argument in favour of some real luminosity effect.

under consideration (according to Pettit and Nicholson the mean for 10 variables : $\Delta m_{bol} = 0^m.9$ and $\Delta m_{vis} = 5^m.3$).* The difficulty in the derivation of the theoretical visual amplitude depends on the inaccuracy of the observed T max. and T min., on the necessity of fixing the effective wave-length of the visual radiation, on the assumption of black-body radiation and so on. To a large degree the question is reduced to the finding of the bolometric corrections for T max. and T min. But at such low temperatures a small error in T is connected with a large change in the bolometric correction. Assuming T max. $= 2300$ and T min. $= 1650$ (Pettit and Nicholson ; Russell, Dugan and Stewart, Table XXX), we find according to (6) the visual amplitude to be equal to $4^m.5$. Perhaps this is to be somewhat increased since the maximum sensitivity of the eye with decreasing brightness is displaced to the shorter wave-lengths. The differential effect of the bands between maximum and minimum, at least in this case, seems to be of the order $1^m.5$. Though for the reasons mentioned above the accuracy of this value must not be overstressed, nevertheless it agrees in general with our preceding results. In any case perhaps there is no need for a large band effect (2^m and more) to account for the large difference between the bolometric and visual amplitudes. It is to be emphasized that the band effect has a definite limit connected with the maximum of the number of molecules *TiO* near $T = 2000°$. The new radiometric measurements by Pettit and Nicholson † and the derived values of T max. and T min. lead to the decrease of the rôle of the bolometric correction and to a considerable increase of the band effect (about 2^m). But in this case for the individual stars we have very discordant results : in a few cases the change in the bolometric corrections is sufficient to account for the large difference between the radiometric and visual amplitude.

The following data derived by Joy for *o* Ceti are instructive with respect to the relative rôle of the bolometric correction and of the total band absorption : ‡ the change in type from M5·2 to M6·3 and from M8 to M9 is connected with a decrease in magnitude of 0·6 and 2·5 respectively, while in the latter case the increase of the strength of bands at the temperature near the minimum seems to be doubtful. Further, it is known that the visual amplitudes of the variables belonging to the later subdivisions of M are much larger than those belonging to the early ones, while in the former case the possible range of the total band effect seems to be smaller.

Assuming even 2^m for the band effect we are not able to explain the large visual amplitude, for instance for χ Cygni, unless the difference of the bolometric corrections between max. and min. is sufficiently large. In addition, the variables of type Se have in the mean the visual amplitude larger than those of Me, while the band effect of Z_rO is much smaller (we consider here the Se stars where the bands *TiO* are either absent or faint).

§ 14. In connection with the effect in question the long-period variables of type μ Cephei are to be mentioned. One may think that in these stars there are small variations in the surface conditions of regular and irregular

* *Ap. J.*, **68**, 279, 1928.
† *Ibid.*, **78**, 320, 1933.
‡ *Ibid.*, **63**, 290, 1926.

character. Since the formation of molecules *TiO* is very sensitive to the change in the temperature, a variation in the strength of bands *TiO* and therefore in the brightness of the star may be expected. But in the light of the results of this paper no large effect is to be expected, unless the variation in the spectrum and in the temperature is sufficiently large (the change from M2 to M5 produces a loss of about $0^m.6$). In this respect it is of interest to secure an accurate spectrophotometric analysis of a variable of μ Cephei type in different phases in order to see whether the variations in the brightness, temperature and perhaps in the diameter are connected with the variations in the strength of the bands *TiO*. Besides, we must keep in mind that at such low temperatures a small change in T produces a considerable change in the bolometric correction and therefore in the visual brightness. However, in the problems of the stars of type M the consequences of the blanketing effect remain quite obscure.

The line and band absorption effect may also play a certain rôle in other problems not considered here. For instance, the value of the slope of the giant and dwarf branch in the Hertzsprung-Russell diagram is influenced by the effect in question. The same applies to the mass-luminosity relation. In the eclipsing variables the spectrum of the secondary is often of much later type than that of the primary, and therefore the observed light ratio L_1/L_2 and other elements are affected perhaps to a small degree by the phenomenon in question.

§ 15. The energy dammed back in the spectral lines and bands must be returned to the layers beneath. This radiation maintains the layers beneath at a temperature higher than they would otherwise possess (blanketing effect). A consequence of this is that the rate of decrease of the intensity from the centre to the limb is reduced. Milne's analysis allows us to calculate this effect in a simple way.* Assuming Planck's distribution of the energy in stellar spectra we can on the basis of the data of fig. 1 compute the total energy missing in lines and the coefficient of the darkening to the limb u (in the case of absence of lines $u_2 = 0.60$).

Table V

	The Total Energy Missing in Lines and Bands	The Mean Coefficient of the Darkening to the Limb
α Lyræ Ao	0.05_8	0.57_2
α Can Min dF5	$.10$	$.55$
Sun (Moon) dGo	$.13_7$	$.53_2$
α Bootis gKo	$.15_3$	$.52_4$
α Tauri gK8	$.17_8$	$.51_1$
β Androm gMo	$.19$	$.50_6$
α Orionis cM2	$.26_5$	$.46_6$
α Scorpii cM2	$.25_4$	$.47_2$
β Pegasi gM2	$.24_5$	$.47_6$
α Herc gM5	$.45_2$	$.36$

* *The Observatory*, **51**, 88, 1928.

As is seen from Table V, the value of the total line absorption increases rather slowly with advancing type down to Mo: a consequence of the decreasing rôle of the violet region (rich in strong lines) with the decreasing temperature. The value 0·15 found by Minnaert in 1927 is somewhat large, for the reasons mentioned in the beginning of this paper. It is of interest to note that Milne derived from Abbot's observations the coefficient of the darkening to be 0·54. This is also near to Lindblad's value.* However, as is evident from Table V, the variation of the mean coefficient of the darkening is not very sensitive to the variation of the total energy missing in lines. The problem of the darkening to the limb for separate wave-lengths is more complicated. This is discussed in Milne's paper and there is no need to repeat it. The effect in question may be of value in connection with the determinations of the orbits of eclipsing binaries and the interferometer measurements of the diameters of stars.

§ 16. Fig. 1 displays a very marked decrease of the total line absorption in the spectral range F5–K8 with increasing wave-length. In the spectra of M-type stars this is greatly complicated by the presence of bands of *TiO*. It raises a question, whether this phenomenon is due to the decrease of the dispersion with the increasing wave-length. It seems to be probable that a systematic error of such kind is inevitable, but we are not able to determine it. However, the evidence mentioned in § 1 tests the reliability of our measurements. We may be more or less sure that the measured total line absorption is not affected by a considerable systematic error. For this reason the reality of the decrease of the total line absorption with increasing wave-length is beyond doubt. This is evidently a characteristic feature of stellar spectra, excepting M-type stars where the molecular bands complicate the phenomenon.

It is of interest to note here an analogy between the distribution of the total line absorption in the stellar spectra and the distribution of the laboratory emission lines along the spectrum.† On fig. 1 at the top is given such a distribution of the frequency of emission lines (the number of lines per 50 A. according to Kayser's *Hauptlinien* . . .). We have for the emission lines almost a fivefold decrease of the number of the laboratory emission lines about 6500 A. as compared with the region about 4000 A. Since in general there is some correspondence between the strength of the emission and absorption lines it is necessary for a full comparison to know the distribution in strength of the laboratory emission lines with respect to wave-lengths. For this purpose we have used Kayser's *Hauptlinien* . . . and Russell's list of ultimate and penultimate lines.‡ However, we must bear in mind that the intensity estimates of the laboratory emission lines are of too arbitrary and indefinite a character. Actually, such a study brings out a small relative increase of the faint lines with increasing wave-length (and so a decrease of the absolute total intensity). This must in general enhance

* *Nova Acta Reg. Soc. Sci. Upsala IV*, vol. vi, No. 1, 1923.

† The same for the solar spectrum was mentioned by Babcock (*P.A.S.P.*, **42**, 209, 1930).

‡ *Ap. J.*, 56, 223, 1925.

the decrease of the total line absorption with increasing wave-length in the absorption spectrum.

The behaviour of fig. 1 and the analogy between the distribution of the laboratory emission lines and the total line absorption in stellar spectra may be probably accounted for by the nature of atomic processes of absorption and emission itself. Actually, at the mean conditions in the star's atmosphere the lines originating from the deepest levels of atoms for well-known reasons are mostly strong, and besides have the higher frequency. On the contrary the lines originating from the upper levels in the average must have the lower frequency, and in addition they are faint. This is true at least for many elements in late type stars. The possible dependence of the general absorption coefficient on wave-length may play a certain rôle in the phenomenon in question, but this will be discussed elsewhere.

Summary

The total line and band absorption for ten stars within classes A0–M5 has been measured and its effect for the photographic and visual regions investigated.

The observed constancy of C.I. or even the decrease of C.I. in several systems when advancing from K5 to M5 and later is wholly accounted for by the total line and band absorption. The dependence of C.I. on the strength of absorption lines is emphasised. The effect of total line absorption is of particular interest for the extra-galactic nebulæ. The total line and band absorption accounts for some discrepancy in spectrophotometric and spectrobolometric measurements, mainly connected with the depression of the continuous background in the violet.

The *apparent* luminosity effect due to the line and band absorption is discussed for M-type stars, especially for the long-period variables (the spectral sub-class as a criterion of absolute magnitude, the spectrum-period-luminosity correlation, the difference between the visual and radiometric amplitudes) : this reaches about $0^m.8$–$1^m.0$ for M5 and about $1^m.7$ for M8e. An attempt is made to derive the total line and band absorption effect from the radiometric measurements.

The effect in question is considered in connection with the phenomenon of the darkening to the limb. The decrease of the total line absorption with the increasing wave-length and an analogous effect for the laboratory emission lines is discussed.

It is a pleasure to thank Professors B. P. Gerasimovič and V. A. Ambarzumian for discussions and criticisms.

Simeis.

INTENSITIES OF INTERSTELLAR LINES.

C. S. Beals.

In this paper are presented measurements of the intensities of interstellar calcium and sodium lines in a number of early-type stars. These measurements represent preliminary results of an extended programme of observation in which it is hoped eventually to include a reasonably large proportion of the northern stars in whose spectra interstellar lines are known to occur.

The instrumental equipment and spectrophotometric methods are for the most part the same as described in an earlier paper * dealing with interstellar intensities in the two stars ξ Persei (H.D. 24912) and 9 Camelopardalis (H.D. 30614). As in the previous paper, the short focus camera (42 cm. focus) with one prism was mainly used in the violet, while the medium focus camera (71 cm. focus) with three prisms was ordinarily used in the visual region, though occasionally arrangements giving greater dispersion were used for the brighter stars. Many of the spectra were taken with a wide slit, which makes possible shorter exposures and the extension of the observations to fainter stars. Since total intensities are involved, this procedure should result in no loss of accuracy. Indeed, in the case of very intense lines an appreciable gain in accuracy is likely to result, both from the larger area measured and from the circumstance that the range of intensities is likely to fall on a more accurately determined portion of the characteristic curve.

All results are given in terms of total intensity as represented by the integral $\int_{\lambda_2}^{\lambda_1}(1-r)d\lambda$,† where r is the ratio of the intensity at any point in the line to that of the neighbouring continuous spectrum, while λ_1 and λ_2 are the displacements from the centre, in angstrom units, corresponding to the extreme edges of the line. Each determination of intensity involved the plotting of a line profile and the measurement of its area by means of a planimeter. With the dispersions which have been used the form of the measured profile is determined almost entirely by the properties of the spectrograph, and no attempt has been made to arrive at the real form of the interstellar line.

Sources of Error

There are a number of well-known sources of error in all photographic spectrophotometry, most of which have been more or less thoroughly discussed by previous investigators. The scattering of light in the photographic emulsion and the contraction of strongly exposed areas of the plate

* *M.N.*, 93, 585, 1933.

† It will be noted that different units of intensity were used in the previous paper.

are probably always present in some degree and are difficult to allow for. Fortunately their effect is small and unlikely to lead to errors of an appreciable amount. The Eberhard or neighbourhood development effect can lead to larger errors, and in the present work efforts have been made to minimise it both by reasonably complete development and by rapid agitation of the solution while development is in progress. For this purpose a special thermostatically controlled development tank has been constructed which is of sufficient length and depth to allow for rapid motion of the developer over the plate when the tank is rocked. Every effort has been made to keep development conditions as uniform as possible, so that any systematic effects due to the above-mentioned causes will affect all plates in a similar way and so minimise errors in the intercomparison of results. Probably the most serious source of accidental error is lack of uniformity in the photographic emulsion. The existence of this lack of uniformity is obvious from an examination of any careful spectrophotometric measure of a stellar spectrogram. It is more troublesome in the faster emulsions, and the areas affected vary from 0·01 mm. (the size of a silver grain) to areas 0·1 mm. or more in linear dimensions. Errors due to this cause are important in the measurement of the fine interstellar lines as it may well happen that an area of non-uniformity is of the same order of size as the area occupied by the line itself. The only means of overcoming the difficulty is to secure sufficient observations to average out the irregularities. For the dispersion used in the present investigation five plates for each line measured has been set up as the standard to be attained, although in the case of plates of good quality reasonably reliable results may be obtained from a smaller number.

The calibration of the plates has been carried out by means of an absorbing wedge placed in front of the slit of an auxiliary spectrograph. Methods of calibrating the wedge and a discussion of the errors involved have been included in another publication.* The accidental errors in the determination of the wedge constant were all between 1 and 2 per cent., and these would correspond to errors of from 2 to 4 per cent. in the measurement of the central intensities of the stronger lines, while the effect on the total intensities will be considerably smaller. Every effort was made to eliminate systematic errors in the wedge calibration, and while it is too much to hope that such efforts have been completely successful, there seems good reason to believe that the systematic errors are too small seriously to affect the results of the present study. A systematic error in the wedge calibration is likely to have its greatest effect in the intercomparison of the lines of sodium and calcium which are located in widely separated parts of the spectrum. Comparison of the same line from star to star will be less affected, as will the comparison of relative intensities in the doublets of sodium and calcium.

The characteristics of the microphotometer which has been used in the measures have been described in the paper just quoted. There seems no reason to doubt the accuracy of performance of the instrument as readings can be reproduced to within less than 1 per cent., and the sensitivity is

* *Pub. D.A.O.*, **6**, 95, 1933.

ample for the measurement of any difference of intensity which can be accurately recorded by the photographic plate. The instrument has been used with a slit width of 0·1 mm. or less, and since the magnification is × 17 this corresponds to a distance on the photographic plate of 0·006 mm. As the total width of an absorption line may be as small as 0·08 mm., there will be an averaging effect, resulting in a distortion of the line profile. The amount of this averaging effect can be calculated by means of an approximation formula due to Rayleigh,* and the computation has been carried out for the profiles in a number of typical cases. It shows itself chiefly in a spuriously large value of r in the neighbourhood of the line centre where the curvature of the profile is greatest. This acts to make the value of the total intensity somewhat too small. The amount of the effect, of the order of 1 per cent., is scarcely appreciable in comparison with errors due to other causes and has been neglected in the final results.

In addition to imperfect photometric technique and the limitations of the photographic plate the existence of stellar absorption blended with the interstellar lines is always a source of possible error. It is well known that stellar calcium behaves irregularly in early-type stars. Struve † has given reasons for expecting lines of stellar calcium to appear occasionally in stars of as early spectral type as Bo, while Plaskett and Pearce ‡ have observed a number of stars of class Bo including H.D. 24534, H.D. 34921 and H.D. 180968, in which stellar Ca^+ lines are certainly present. On the other hand, the writer has been informed by Mr. W. E. Harper that there are occasional Ao stars in which stellar calcium does not appear. In the present investigation the writer has detected displaced stellar lines almost certainly due to calcium and sodium in the star P Cygni, which is of type B1 and, as is well known, has its absorption lines strongly displaced to the violet. The star ρ Leonis, of type Bo, shows a relatively broad diffuse line at λ 3933, presumably due to stellar calcium, with a narrow interstellar line superposed upon it. There is no evidence of the presence of stellar sodium in this star. In fig. 1 are shown line profiles from the spectra of P Cygni and ρ Leonis, which have indicated the presence of the stellar lines. No observations are available of stellar calcium or sodium in the spectra of stars of types O8 or O9, but it seems possible, or even probable, that stellar calcium may sometimes be present blended with the interstellar lines.

Numerical Results

The results of the measures of total intensity are included in Table I. The first two columns give the H.D. number and the name or other designation of the star. The third column gives the spectral type of the star according to Plaskett and Pearce. The fourth column gives the mean value of the total intensity of λ 3933 of *Ca*, followed by the calculated probable error. The fifth column gives the number of plates on which the line has been

* Rayleigh, *Scientific Papers*, **1**, 135, 1871.
† *Ap. J.*, **67**, 353, 1928.
‡ *Pub. D.A.O.*, **5**, 167, 1933.

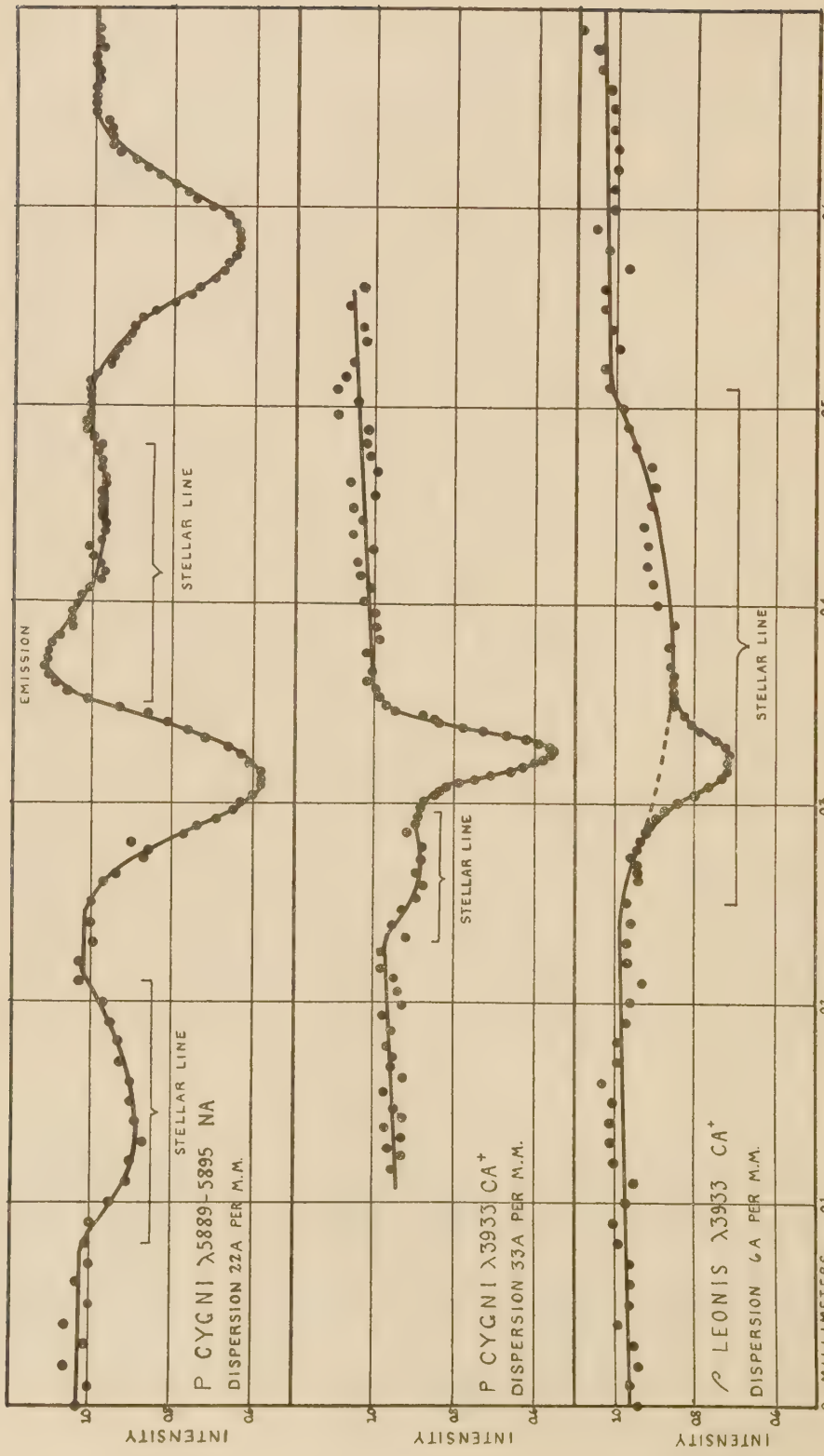

P CYGNI λ5889-5895 NA
DISPERSION 22A PER M.M.
EMISSION
STELLAR LINE
STELLAR LINE
P CYGNI λ3933 CA+
DISPERSION 33A PER M.M.
STELLAR LINE
ρ LEONIS λ3933 CA+
DISPERSION 6A PER M.M.
STELLAR LINE
INTENSITY
INTENSITY
INTENSITY
1.0
0.8
0.6
0
0.1
0.2
0.3
0.4
0.5
0.7
MILLIMETERS

FIG. 1.

measured. Similar information concerning the line λ 3968 of *Ca* is given in columns 6 and 7. The next two columns give the intensities of the sodium lines, while the tenth column gives the number of plates on which the sodium doublet has been measured.

TABLE I

Intensities of Interstellar Lines

H.D.	Desig.	Type	I (λ 3933)	No.	I (λ 3968)	No.	I (λ 5889)	I (λ 5895)	No.
13841	56° 470	B2sk	·726 ± ·013	5					
13854	Boss 507	B0sk	·796 ± ·035	5					
14134	Boss 519	B2sk	·923 ± ·017	5					
14143	56° 530	B1sk	·910 ± ·039	4					
14818	10 Persei	B1sk	·801 ± ·014	6			1·086	·794	2
14956	57° 568	B1sk	·682 ± ·041	5					
24912	ξ Persei	O7nk	·212 ± ·018	5			·371 ± ·026	·295 ± ·014	8
30614	9 Camel	O9sek	·374 ± ·033	5			·654 ± ·043	·538 ± ·031	7
36861	λ Orion	O8sk	·219 ± ·017	5			·346 ± ·014	·263 ± ·010	6
37043	ι Orion	O8sk	·056	2			·257 ± ·018	·173 ± ·020	3
47839	S Monoc	O7sk	·237 ± ·031	7	·127	2			
91316	ρ Leonis	B0sk	·045	2	·029	1	·210 ± ·010	·104 ± ·012	6
90429N	35° 3930N	O5nfk	·578 ± ·029	3					
90429S	35° 3930S	O9nk	·543	1					
190918	35° 3953	O9kW5	·592	1	·266	1			
191765	35° 4001	W6	·627	1	·258	1			
192163	37° 3821	W6	·402	2	·411	2			
192639	36° 3958	O7sfk	·540	1					
193237	P Cygni	B1qk	·458 ± ·013	6			·481 ± ·022	·523 ± ·012	6
203064	A Cygni	O8nk	·342 ± ·038	6	·269 ± ·005	4	·395	·336	1
210839	λ Cephei	O6nek	·343 ± ·024	8	·318 ± ·033	4	·748 ± ·020	·666 ± ·022	7

The line λ 3968 of calcium was measured on relatively few plates as the proximity of the hydrogen line *Hε* made an accurate determination of its intensity very difficult. Even where the breadth of *Hε* made it possible to draw in the background, a considerable amount of uncertainty still remains, and in discussions of the relative intensities of doublet members main reliance has been placed in the sodium lines. The Wolf-Rayet stars offer what appears to be the best opportunity for obtaining accurate values of relative intensity in the calcium doublet, but the observations so far obtained are too fragmentary to yield definite results.

Intensities in Perseus Cluster

In any investigation of interstellar lines the comparison of intensities in a close physical cluster like *h* and *χ* of Perseus becomes of considerable interest and importance. If the general conclusions drawn by Plaskett and Pearce * as to the relation between distance and intensity are valid, then stars in a given cluster should show a close similarity in the intensities of

* *Pub. D.A.O.*, 5, 167, 1933.

their interstellar lines. The six stars from the Perseus Cluster (H.D. 13841, 13854, 14134, 14143, 14818 and 14956) listed in Table I are alike in that they all show very intense interstellar lines, but the differences from star to star are in some cases several times the probable errors and there seems comparatively little doubt that they are real. The following suggestions may be put forward to explain these differences :—

(a) *On the assumption of uniform distribution of interstellar matter the differences in intensity may be a result of the observation of foreground or background stars not actually members of the cluster.* This hypothesis is difficult to disprove completely, but the community of motion shown by the stars in question make it highly probable that they are all members of the same physical cluster.

(b) *The differences may be due to lack of uniformity in the distribution of interstellar calcium.* If the stars were enveloped in a cloud of calcium of unusually great density, of the same order of size as the cluster itself, then the differences in distance due to the linear dimensions of the cluster might account for the variations among the interstellar lines. Or, as suggested by Young,* separate stars might be surrounded by clouds of calcium of differing optical depths. In view of the general evidence adduced by Plaskett and Pearce concerning the uniformity of distribution of interstellar calcium the above explanations are unlikely to apply. In particular, the well-marked difference between stellar and interstellar velocity which is shown with great regularity by each member of the group is strong evidence against the hypothesis of local surrounding clouds.

(c) *The differences in intensity are due to blends with stellar calcium.* It has already been pointed out that stellar calcium is sometimes present in appreciable intensity in stars as early as B1 and Bo, and since the spectral types of the stars in question range from Bo to B2 the existence of blends with stellar calcium is very probably the main cause of the differences in intensity. As has already been pointed out, stellar calcium behaves very irregularly in the hotter stars so that no clear correlation with spectral type would be expected. It may be seen from fig. 1 that in the case of the star ρ Leonis, of spectral type Bo, the stellar line is considerably stronger than the interstellar. Moreover, both in the spectrum of this star and of the star P Cygni the total intensity of the stellar lines is quite sufficient to account for differences of intensity such as have been observed in the Perseus cluster.

Relative Intensities in Perseus and Cygnus Clusters

In Table I may be found measures of the K line in a number of stars (H.D. 190429N, H.D. 190429S, H.D. 190918, H.D. 191765, H.D. 192163, H.D. 192639) belonging to a small group in Cygnus which appears to be somewhat analogous to the cluster in Perseus previously discussed. The observations of these stars are as yet very incomplete, but the number of measurements available is sufficient to make possible a general comparison between the intensities of interstellar lines in the two clusters. The mean

* *Pub. D.A.O.*, 1, 219, 1920.

value for the Perseus cluster is 0·806 and that for the Cygnus stars is 0·546 in qualitative agreement with previous estimates by Plaskett and Pearce.* The difference in intensity is generally interpreted as due to the greater distance of the Perseus cluster in spite of the fact that the apparent magnitudes of the Cygnus stars are on the average 0·6 mag. fainter than those in Perseus. The true explanation of this discrepancy is not known with certainty. There seems little probability that it can be due to any lack of uniformity in the distribution of interstellar matter. The sharp-lined character of the spectra of the stars in the Perseus cluster suggests that they are of unusually high intrinsic luminosity, and this may well be the explanation of their greater apparent brightness. There is also the possibility that the apparent brightness of the Cygnus stars may be affected in some degree by the presence of a cloud of obscuring matter unrelated to the interstellar gas.

Relative Intensities of Interstellar Calcium and Sodium Lines

The results in Table I serve, in general, to confirm the earlier values derived for the stars ξ Persei and 9 Camelopardalis and to indicate that, on the basis of the intensity units employed in Table I, the sodium lines are considerably stronger than those of calcium. The disparity in intensity, as might be expected, is greater for stars with relatively weak lines (*i.e.* ρ Leonis and ι Orionis). For the stars P. Cygni and H.D. 14818 the lines of calcium are stronger relative to sodium than in other stars of Table I. In the P Cygni spectrum this apparent discrepancy probably finds its explanation in the existence of weak emission lines of stellar sodium superposed on the strong interstellar absorption. The relative intensities in the spectrum of H.D. 14818 are more difficult to explain. They may be simply an effect of accidental error as only two plates are available for measures of the sodium lines. It should be pointed out, however, that these plates are of excellent quality and give accordant values of the intensity, so that it is doubtful whether additional observations will substantially alter the results. It is possible, though scarcely probable, that the disparity in intensity is produced by a systematic error in the wedge constant. (If the value for λ 3933 were too large the effect would show itself more strongly in the case of stronger lines such as those of the Perseus cluster.) Finally, and this is probably the most plausible explanation, the greater intensity of the calcium line may be due to a blend with stellar calcium, the sodium lines being free from any similar blending effects.

Relative Intensities of Members of the Sodium Doublet

In considering the ratio of intensity of the two lines of a doublet main reliance will be placed on measures of the sodium lines, since the proximity of $H\epsilon$ to the line λ 3968 of calcium makes it difficult to secure accurate estimates of intensity. Moreover, it seems probable that in general the sodium lines will be less affected by blends with stellar absorption lines. In

* *Pub. D.A.O.*, 5, 167, 1933.

the spectrum of ρ Leonis, as has already been mentioned, stellar and interstellar calcium are both present, and the sharp interstellar line can be detected superposed on the broader stellar absorption. There is no evidence of stellar absorption due to sodium in the spectrum of this star, though the dispersion used should be adequate to detect it if it were present in any strength. Stellar sodium is present in P Cygni, but the spectrum of this star is very unusual and possibly not typical of others of approximately the same spectral class. In the discussion which follows it will be assumed that stellar sodium is not present as a complicating factor, though it is realised that the assumption may be in error to a greater or less degree.

TABLE II

Ratio of Doublet Intensities

Star	(5889)	(5895)	Ratio $\frac{5889}{5895}$	No. Plates
ρ Leonis	·210	·104	2·02	6
ι Orionis	·257	·173	1·49 *	3
λ Orionis	·346	·263	1·31	6
ξ Persei	·371	·295	1·26	8
A Cygni	·395	·336	1·18	1
P Cygni	·481	·523	0·92 †	6
9 Camel	·654	·538	1·21	7
λ Cephei	·748	·666	1·12	7
H.D. 14818	1·086	·794	1·37	2

* There is some evidence of blends in the case of these lines.
† This ratio is probably affected by blends with stellar emission.

Table II gives the relation between the intensity ratio 5889/5895 and the total intensities of the two lines. The ratio is approximately 2·0 for mean intensity 0·157 in ρ Leonis, decreases to approximately 1·2 in the range of intensity from 0·400 to 0·700 and then increases to 1·37 for intensity 0·940 in the star H.D. 14818 which has the strongest interstellar lines of any star so far measured.

In view of the meagreness of the data, considerable caution must be exercised in endeavouring to interpret the results so that it is scarcely possible at present to do more than suggest certain interesting possibilities. Unsöld, Struve and Elvey * have worked out quantitatively the variation to be expected in the ratio of total intensities of doublet members when the form of the scattering coefficient is determined partly by thermal Doppler effect and partly by radiation damping. They treated the case of a line with a perfectly flat Doppler effect core corresponding to velocities of thermal agitation of 5 km./sec. with "wings" of varying width due to the effect of radiation damping. For a layer of gas of small optical depth, Doppler effect will predominate and the intensity ratio will approach the value of 2·0

* *Zs. f. Astrophysik*, **1**, 314, 1930.

corresponding to the ratio of "oscillator strengths" of the two lines.* As the optical depth of the layer increases, absorption becomes more and more complete and the ratio of total intensities tends to approach unity. This tendency is checked by the appearance of wings on the lines due to the influence of radiation damping which becomes more important as the optical depth becomes greater. The ratio of total intensities accordingly increases again and finally approaches the value of $\sqrt{2}$ corresponding to a mechanism of pure radiation damping.

It will be noted that the results of these calculations are in fair accord with the observed variation in the ratio of total intensities of members of the sodium doublet, and while the assumptions can scarcely correspond precisely to physical conditions existing in interstellar space, the general line of argument appears to lead in the right direction. For convenience in calculation Unsöld, Struve and Elvey assumed a velocity of thermal agitation of 5 km./sec., all atoms having the same velocity. Actually the mean velocity of thermal agitation for sodium atoms at the temperature of 10,000° derived by Eddington for interstellar space is approximately 3 km./sec., and the distribution of velocities will presumably correspond to a Gaussian error curve. Such a distribution will lead to a scattering coefficient relatively broad and extended in form, so that for a layer of gas of small optical depth the ratio of doublet intensities should approach the value 2 : 1 to a first approximation at least. The subsequent decrease in the value of the intensity ratio may be expected to be less marked than for a scattering coefficient perfectly flat in form owing to the wings produced by the spread in velocities ; and the minimum value observed for this ratio is 1·12 for the star λ Cephei. In the absence of other complicating factors the ratio of intensities for the stronger lines should be governed almost wholly by radiation damping, and it is rather striking to find the ratio 1·37 in the star H.D. 14818, which agrees within the probable error of measurement with the predicted value $\sqrt{2}$. While this agreement appears at first glance to be remarkably satisfactory, it should perhaps not be taken too seriously, since Doppler effect due to galactic rotation is almost certain to enter as a complicating factor for stars as distant as those in the Perseus cluster. Fortunately the variation of the galactic rotation effect with galactic longitude offers a possibility for making a definitive test of this point when sufficient new observations become available.

Summary and Conclusion

Measures of the total intensities of interstellar lines in a number of early-type stars in the h and χ cluster in Perseus have indicated differences in intensity which can scarcely be due to differences in distances of the stars. The most reasonable explanation of these effects seems to be that they are due to blends with stellar calcium.

The measures have confirmed the observations of earlier workers that interstellar lines in the Perseus cluster are stronger than those in a corresponding cluster in the Cygnus region.

* See discussion by Struve and Elvey, *Ap. J.*, **72**, 267, 1930.

Earlier results by the writer on the relative intensities of sodium and calcium lines are confirmed, and it is shown that the sodium lines are considerably stronger than those of calcium. Exceptions to this rule may be explained on the basis of blends with stellar lines.

An attempt has been made to interpret the relative intensities of members of the sodium doublet in terms of the atomic mechanism producing the absorption. The results are shown to be consistent with the hypothesis that thermal Doppler effects are the main determining influence on the scattering coefficient for relatively faint lines, but that combined radiation damping and Doppler effects due to galactic rotation probably predominate in lines of greater intensity. The variation of the galactic rotation effect with galactic longitude is suggested as the logical avenue of approach in making further investigations of the problem.

In conclusion I wish to thank Mr. S. N. Hill, of the staff of this observatory, for valuable assistance in the measurement and reduction of plates.

Dominion Astrophysical Observatory,
Victoria, B.C. :
1934 *March* 20.

ON PERIODIC OR SECULAR CHANGES OF VELOCITY ON JUPITER. (Third Paper.)

A. Stanley Williams.

It is proposed to consider in this paper * the whole surface of the planet south of the great Equatorial Current, excepting that part which includes the bright S. Tropical Zone and the southern half of the (usually double) S. Equatorial Belt.†

(3) *The South Temperate Current*

For present purposes this current is deemed to include the dark S. Temperate Belt and a considerable region to the south of that belt. The approximate limits in latitude are from 27° S. lat. to 37° S. lat. But the southern limit is sometimes somewhat indefinite, and its latitude may vary a little from year to year, or even occasionally in the same year in different longitudes of the planet. The surface material farther south (forming the Southern Current) has a swifter motion than that of the S. Temperate Current, but the line of demarcation between the two currents is apparently

* See *M.N.*, 93, 409, and 94, 240, for the two previous papers on this subject.

† Excepted because of the complications occasioned by the presence in the same latitude of the Red Spot and the S. Tropical Disturbance with their different and variable motions and changes.

not always well marked or defined. Both these two currents are, however, distinct and permanent features of the planet, and they are always observable.

Table I gives the observations of the rotation period of the S. Temperate Current in the different years in the same form as before. The results subsequent to 1907 have also, as before, been derived from the various *Memoirs* of the Jupiter Section of the British Astronomical Association. The observations with weight 3 are not likely to be in error by more than one second of time and are thoroughly to be relied upon. But the matter of these weights and the accuracy of the observations will be referred to again further on.

TABLE I

Observed Rotation Period of S. Temperate Current

Year	Rotation Period	Wt.	Year	Rotation Period	Wt.
	h m s			h m s	
1787	9 55 17·6	...	1907·8	9 55 20·5	3
1862	17·2	1	1908·9	19·9	3
1872·3	19·6	1	1909·10	19·9	3
1880	16·2	1	1911	12·7	2
1881	18·3	1	1912	20·6	2
1887	17·5	2	1913	22·9	1
1888	[08·2]*	1	1914	20·2	2
1889	18·2	2	1915	20·9	2
1890	17·2	3	1916·7	21·3	2
1890·1	18·3	1	1917·8	18·8	2
1891	19·0	1	1918·9	21·8	2
1898	18·3	3	1919·20	11·9	2
1899	17·9	3	1920·1	16·6	2
1900	16·9	3	1922	15·0	1
1901	18·0	3	1923	14·3	2
1902·3	18·5	3	1924	...	...
1903·4	18·0	3	1925	22·6	2
1904·5	21·0	3	1926	23·1	2
1906·7	21·9	3	1927·8	22·6	3

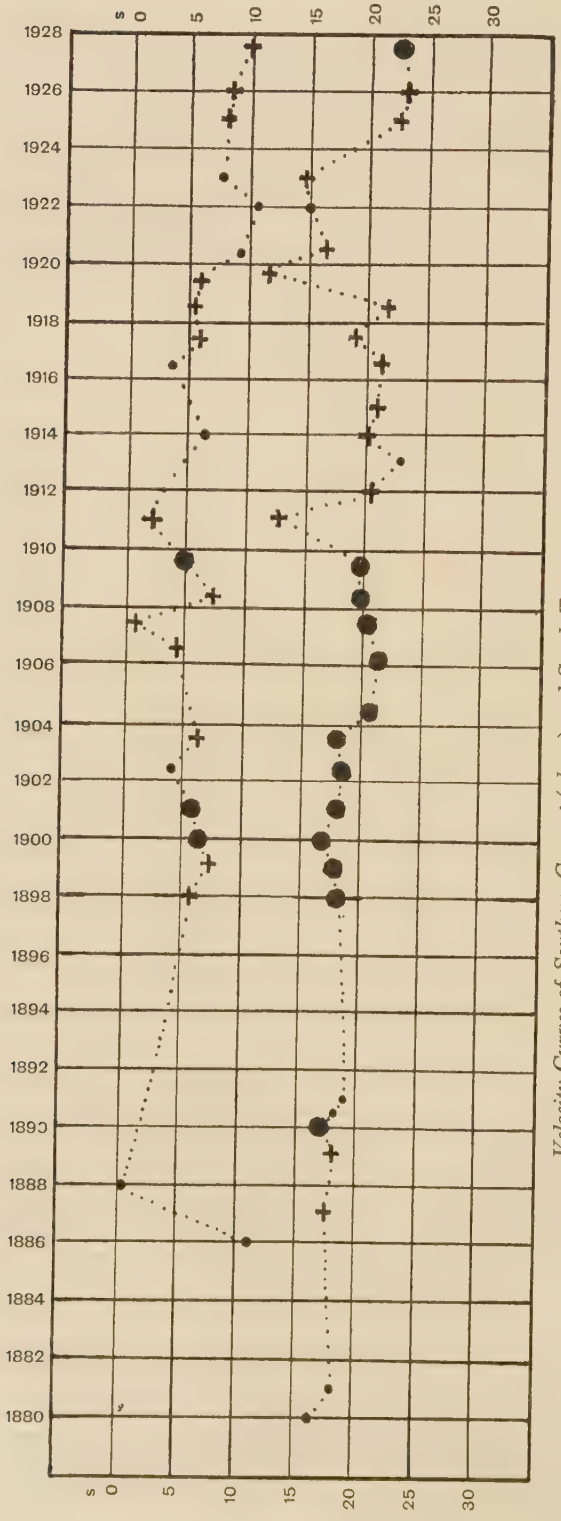
* This was the single abnormal spot described in *M.N.*, **59**, 29, 1898.

The observations are also shown in diagram form in the lower part of the accompanying figure, from which it will be seen that there have only been small changes in the velocity of the current since 1880, and that there is no definite indication of any periodicity. There has, however, been a slow increase in the length of the rotation period. The observations can be satisfied quite as closely as could be expected by a straight line drawn through them, or by the following simple linear formula,

$$R = 9^{h}\ 55^{m}\ 17^{s}\cdot0 + 0^{s}\cdot115\ E,$$

where R = the rotation period * and E the time in years from 1880, the zero epoch. It is interesting to note that the change, slight though it is,

* The rotation period of a spot on Jupiter is the mean solar time in which the spot completes a sidereal rotation. The rotation period of a Jovian current would have the same meaning.



Velocity Curves of Southern Current (above) and South Temperate Current (below).
Vertical scale, $60^s = 62$ km./h (for latitude about 30°). ● = *weight* 3, + = *weight* 2, • = *weight* 1.

has been in the same direction as that of the Equatorial Current. The early observations of 1787 (Schroeter), 1862 (Schmidt) and 1872·3 (Lohse) seem to show that there can hardly have been much change during the 93 years previous to 1880.

The equable nature of the motion of the S. Temperate Current is really remarkable,* and advantage has been taken of it to make an independent estimate of the probable error of a determination of the rotation period of a current on Jupiter of weight 3. There are 13 observations of weight 3 included in Table I, and these reduced to one date by means of the above formula give $\pm 0^s{\cdot}92$ as the mean error of an observation. The statement previously made, that the observations with weight 3 are not likely to be in error by much more than one second of time is thus fully confirmed. Similarly, the mean error of an observation of weight 2 is $\pm 3^s$, but it would be less if two or three of the extreme values were rejected on the ground that some of the spots observed may have been in or affected by their proximity to the more swiftly moving Southern Current, as had indeed been surmised by Phillips.†

It should be observed that the requirements for a weight 3 are somewhat exacting. (1) There must be a dozen (at least) of spots arranged pretty regularly all round the planet. (2) The spots must be well observed. That is, the observations must be numerous and extend over a considerable period of time, several months at least. (3) There must not be any serious uncertainty with regard to identification. This is of the first importance. And lastly, the skill and experience of the observer and computer are also factors to be taken into consideration.

(4) *The Southern Current*

Extending from about lat. 37° S. to about lat. 55° S. The rather indefinite nature of the limit between this current and the S. Temperate Current has been already referred to, and the southern limit also appears to be indefinite or variable in its latitude, though this may be due to the difficulty of observing spots so far south, particularly as regards their latitude. Table II gives the observations of the rotation period of the current, and they are also shown in diagram form in the upper part of the figure.

The accordance of the observations is very satisfactory having regard to the difficulty in making observations of spots so far south. Since 1888 there has been a small and nearly uniform increase in the period of rotation like that already found for the S. Temperate Current, but a slightly curved line would suit the observations better than a straight line. The single spot observed in 1886 was in S. lat. 50° according to the observations made by Young with the 23-inch refractor of the Princeton Observatory, and

* As long ago as 1903 the current had been described by Phillips as being "the steadiest of all the Jovian currents," a description fully confirmed by the subsequent observations.

† See *Mem. B.A.A.*, **27**, 82, 1927.

possibly it should not have been included in the table. It may have been in the more southern current next to be described.

TABLE II

Observed Rotation Period of Southern Current

Year	Rotation Period	Wt.	Year	Rotation Period	Wt.
	h m s			h m s	
1886	9 55 11·1	1	1911	9 55 1·6	2
1888	0·2	1	1914	6·4	1
1898	6·0	2	1916·7	3·5	1
1899	7·2	2	1917·8	6·0	2
1900	6·6	3	1918·9	5·4	2
1901	6·1	3	1919·20	6·0	2
1902·3	4·0	1	1920·1	9·0	1
1903·4	6·3	2	1922	10·7	1
1906·7	4·2	2	1923	7·8	1
1907·8	1·0	2	1925	8·1	2
1908·9	7·3	2	1926	8·4	2
1909·10	5·0	3	1927·8	9·8	2

(5) *The South Polar Current*

South of the Southern Current a decidedly slower motion or longer rotation period is indicated, but owing to the great difficulty in making observations so near the pole the results are few in number and necessarily somewhat discordant. The mean rotation period is $9^h\ 55^m\ 28^s$ from 8 determinations by four observers, Bolton, Hawks, Molesworth and Phillips. The spots or markings observed were on or about the N. edge of the dark S. Polar Cap. The northern limit of this current is in about 55° S. lat. The southern limit is unknown. The names given in these papers to the chief Jovian currents are those suggested in 1898 in *M.N.*, **59**, 33, with some necessary modifications. The present current is an addition to the number. The name had already been used by Phillips.

Summary

(1) The systematic observations of the last half century show that the motion of the S. Temperate Current has been remarkably equable throughout the period, but that there has been a slow but uniform increase in the length of the rotation period amounting to $0^s{\cdot}115$ per annum.

(2) There is no definite indication of any periodicity in the velocity of the current.

(3) Advantage has been taken of the equable motion to make a fresh and independent estimate of the accuracy with which the rotation period of a current on Jupiter can be determined.

(4) The observations of the Southern Current indicate a slow increase in length of rotation period nearly similar to that of the S. Temperate Current.

(5) The equable velocities of the surface material of the planet south of lat. 27° is in striking contrast to the considerable and rapid changes occurring in the equatorial regions of the planet.

Polvarth, St. Mawes :
1934 *May* 3.

Erratum

H. Knox-Shaw, "An Analysis of the Radcliffe Proper Motions in the Selected Areas," p. 399, last line, *for* 0″·5 *read* 0″·05.